STOKE'S THEOREM

Statement:

Let S be an open surface bounded by a closed curve C and vector F be any vector point function having continuous first order partial derivatives. Then

$$\oint_C \vec{F} . \vec{dr} = \iint_S \text{curl } \vec{F} . \check{n} \, ds$$

where ***ň****= unit normal vector at any point of S drawn in the sense in which a right handed screw would advance when rotated in the sense of the description of C.*

Stokes's Theorem relates a surface integral over a surface S to a line integral around the boundary curve of S (a space curve).

Example 1

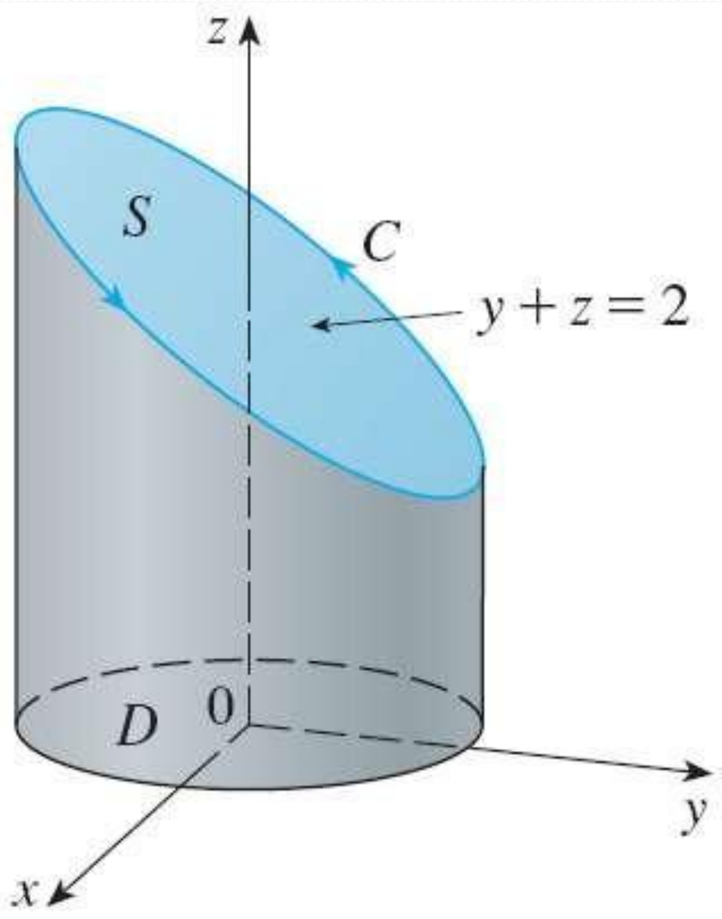
Evaluate

$$\int_C \vec{F} \cdot d\vec{r}$$

where:

- $F(x, y, z) = -y^2\, i + x\, j + z^2\, k$
- *C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$. (Orient C to be counterclockwise when viewed from above.)*

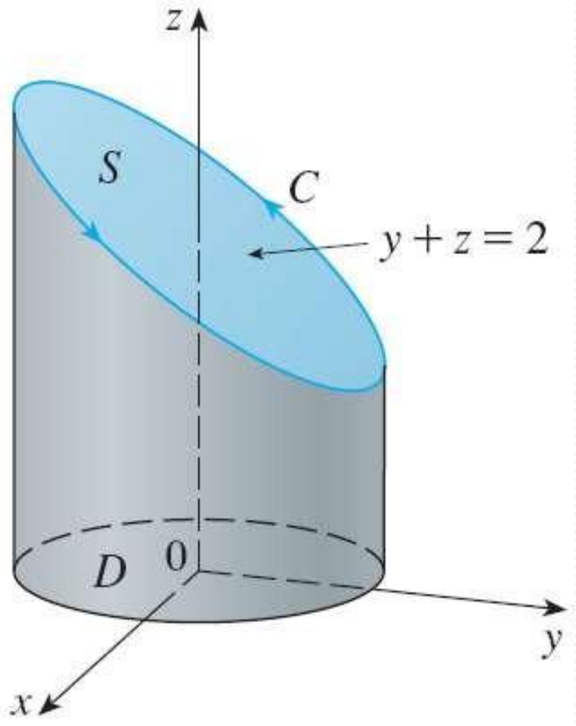
The curve C (an ellipse) is shown here.



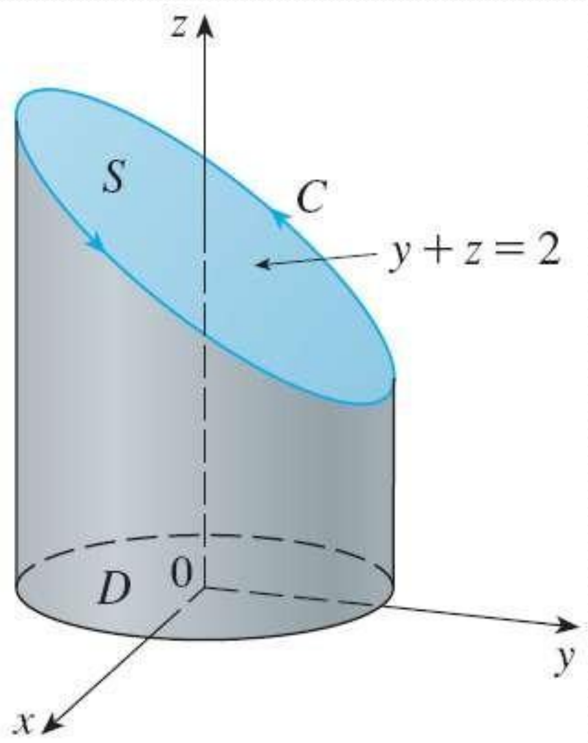
We first compute:

$$\operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{vmatrix} = (1+2y)\mathbf{k}$$

• The elliptical region S in the plane $y + z = 2$ that is bounded by C. If we orient S upward, C has the induced positive orientation.



• The projection *D* of *S* on the *xy*-plane is the disk $x^2 + y^2 \leq 1$.



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D (1 + 2y)\, dA$$

$$= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^2}{2} + 2\frac{r^3}{3} \sin\theta \right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\tfrac{1}{2} + \tfrac{2}{3} \sin\theta \right) d\theta$$

$$= \tfrac{1}{2}(2\pi) + 0 = \pi$$

Example 2

Use Stokes' Theorem to compute $\iint$ **curl F.ds**

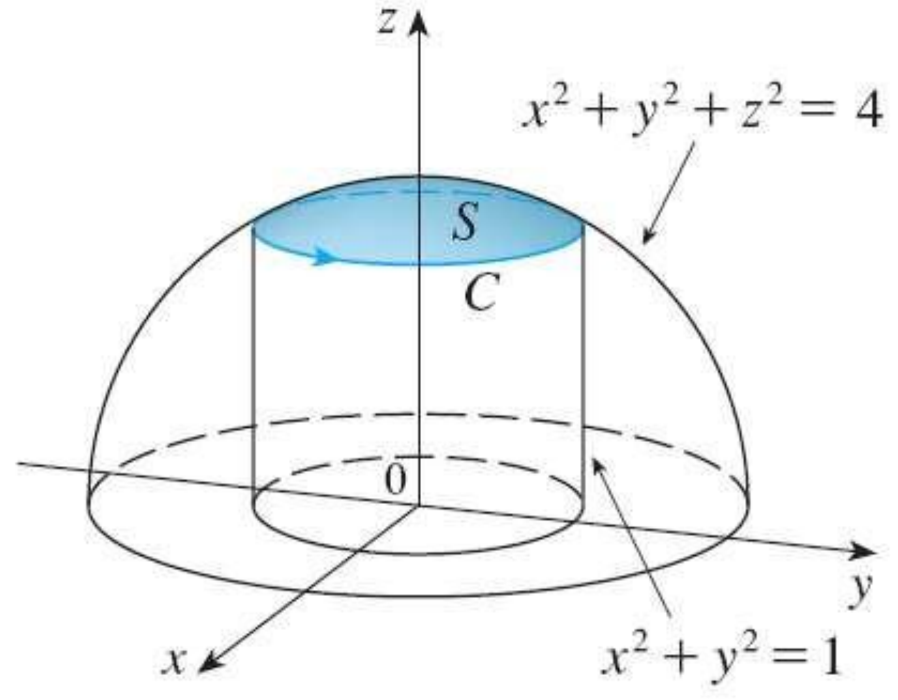
where:

- $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + yz\,\mathbf{j} + xy\,\mathbf{k}$
- S is the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies inside the cylinder $x^2 + y^2 = 1$ and above the xy-plane.

To find the boundary curve *C*, we solve:

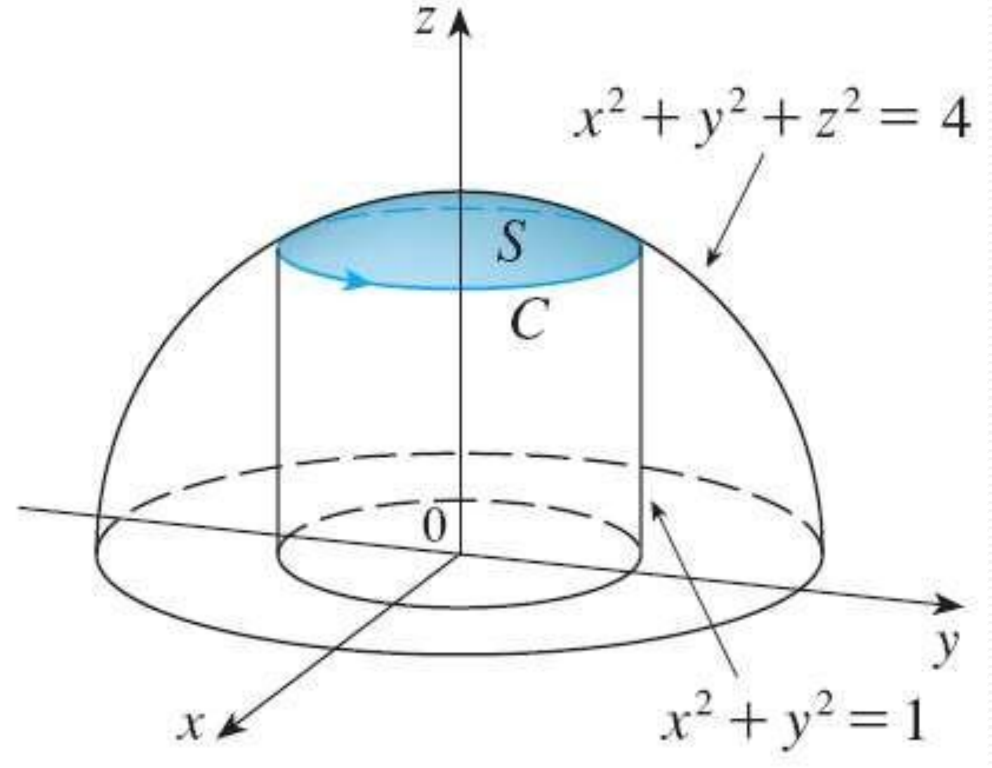
$$x^2 + y^2 + z^2 = 4 \quad \text{and} \quad x^2 + y^2 = 1$$

- Subtracting, we get $z^2 = 3$.
- So, $z=\sqrt{3}$ (since $z > 0$).



So, C is the circle given by:

$$x^2 + y^2 = 1, \; z=\sqrt{3}.$$



A vector equation of C is:

$$\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{3}\mathbf{k} \qquad 0 \le t \le 2\pi$$

Therefore, $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$

Also we have:

$$\mathbf{F}\big(\mathbf{r}(t)\big) = \sqrt{3}\cos t\,\mathbf{i} + \sqrt{3}\sin t\,\mathbf{j} + \cos t \sin t\,\mathbf{k}$$

Thus, by Stokes' Theorem,

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$$

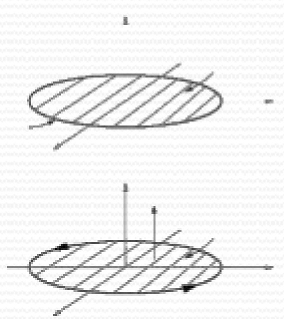
$$= \int_0^{2\pi} \left(-\sqrt{3} \cos t \sin t + \sqrt{3} \sin t \cos t \right) dt$$

$$= \sqrt{3} \int_0^{2\pi} 0\, dt = 0$$

Example 3

Verify Stokes' Theorem for the field F =<x^2, 2x, z^2> on the ellipse S = {(x, y , z) : $4x^2 + y^2$ 6 4, z = 0}.

Solution: We compute both sides in $\int_c F \cdot dr = \iint_s \text{curl } F \cdot n \, d\sigma$.



We start computing the circulation integral on the ellipse $x^2 + \frac{y^2}{2^2} = 1$.

We need to choose a counterclockwise parametrization, hence the normal to S points upwards. We choose, for $t \in [0, 2\pi]$, $r(t) = <\cos(t), 2\sin(t), 0>$.

Therefore, the right-hand rule normal n to S is n =<0, 0, Recall: **$\int_c F \cdot dr = \iint_s \text{curl } F \cdot n\, d\sigma$.** **$r(t) = <\cos(t), 2\sin(t), 0>, t \in [0, 2\pi]$ and $n = <0, 0, 1>$.** The circulation integral is:

$$\int F.\,dr = \int_0^{4\frac{\pi}{2}} F(t) \cdot r'(t)\,dt$$

$$\int_0^{4\frac{\pi}{2}} < \cos(t) * \cos(t), 2\cos(t), 0 > \cdot < -\sin(t), 2\cos(t), 0 > dt.$$

$$\int_c F.\,dr = \int_0^{4\frac{\pi}{2}} \{-\cos(t) * \cos(t)\sin(t) + 4\cos(t) * \cos(t)\}dt.$$

The substitution on the first term $u = \cos(t)$ and $du = -\sin(t)\,dt$ implies

$$\int_0^{4\frac{\pi}{2}} \{4\cos(t) * \cos(t)\}dt. = \int_0^{4\frac{\pi}{2}} \{2(1 + \cos 2t\}dt.$$

Since $= 0 \int_0^{4\frac{\pi}{2}} \{\cos 2t\}dt.$ we conclude that $\int_c F.\,dr = 4\pi$

We now compute the right-hand side in Stokes' Theorem.

$I = \iint_S \text{curl } F \cdot n \, d\sigma.$

Solving, we get: curl $F = <0,0,2>$.

S is the flat surface $\{x^2 + \frac{y^2}{2^2} <= 1, z = 0\}$ so $d\sigma = dx\,dy$.

Then, $\iint (\text{curl } F).n \, d\sigma = \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} < 0,0,2 > . < 0,0,1 > \; dy \, dx$

The right-hand side above is twice the area of the ellipse. Since we know that an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has area πab, we obtain

$$\iint_S \text{curl } F \cdot n \, d\sigma = 4\pi.$$

This verifies Stokes' Theorem.

Thank you!!!!!