UNIT-1

TIME - VARYING FIELDS AND MAXWELL’S EQUATIONS

Time-Varying Fields

Stationary charges $\longrightarrow$ electrostatic fields

Steady currents $\longrightarrow$ magnetostatic fields

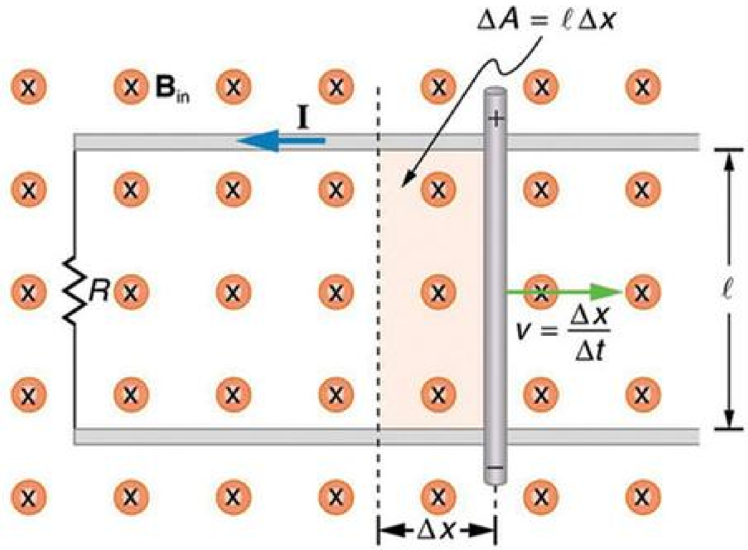
Time-varying currents $\longrightarrow$ electromagnetic fields

Only in a non-time-varying case can electric and magnetic fields be considered as independent of each other. In a time-varying (<u>dynamic</u>) case the two fields are interdependent.

A changing magnetic field induces an electric field, and vice versa.

Motional EMF

- A rod is moved at a speed v along a pair of conducting rails separated by a distance ℓ in a uniform magnetic field B.
- The rails are stationary relative to B
- Connected to a stationary resistor R
- Consider the area enclosed by the moving rod, rails and resistor.
- B is perpendicular to this area, and the area is increasing as the rod moves.



Thus the magnetic flux enclosed by the rails, rod and resistor is increasing.

When flux changes, an EMF is induced according to Faraday's law of induction.

The Continuity Equation

Electric charges may not be created or destroyed (the principle of conservation of charge).

Consider an arbitrary volume *V* bounded by surface *S*. A net charge *Q* exists within this region. If a net current *I* flows across the surface out of this region, the charge in the volume must decrease at a rate that equals the current:

$$I = \int_S \overline{J} \cdot \overline{dS} = -\frac{dQ}{dt} = -\frac{d}{dt}\int_V \rho_V dv$$

Divergence theorem

$$\int_V \nabla \cdot \overline{J} dv = -\int_V \frac{\partial \rho_V}{\partial t} dv$$

Partial derivative because may be a function of both time and space

This equation must hold regardless of the choice of *V*, therefore the integrands must be equal:

$$\boxed{\nabla \cdot \overline{J} = -\frac{\partial \rho_V}{\partial t}} \quad (A/m^3)$$

the equation of continuity

For steady currents

$$\boxed{\nabla \cdot \overline{J} = 0}$$

Kirchhoff's current law

follows from this

that is, steady electric currents are divergences or solenoidal.

Displacement Current

For magnetostatic field, we recall that

$$\nabla \times \overline{H} = \overline{J}$$

Taking the divergence of this equation we have

$$\nabla \cdot \left(\nabla \times \overline{H}\right) = 0 = \nabla \cdot \overline{J}$$

However the continuity equation requires that

$$\nabla \cdot \overline{J} = -\frac{\partial \rho_V}{\partial t} \neq 0$$

Thus we must modify the magnetostatic curl equation to agree with the continuity equation. Let us add a term to the former so that it becomes

$$\nabla \times \overline{H} = \overline{J} + \overline{J}_d$$

where $\overline{J}$ is the conduction current density $\overline{J} = \sigma_E \overline{E}$, and $\overline{J}_d$ is to be determined and defined.

Displacement Current continued

Taking the divergence we have

$$\nabla\cdot\left(\nabla\times\overline{H}\right)=0=\nabla\cdot\overline{J}+\nabla\cdot\overline{J}_d \longrightarrow \nabla\cdot\overline{J}_d=-\nabla\cdot\overline{J}$$

In order for this equation to agree with the continuity equation,

Gauss' law

$$\nabla\cdot\overline{J}_d=-\nabla\cdot\overline{J}=\frac{\partial\rho_V}{\partial t}=\frac{\partial}{\partial t}\left(\nabla\cdot\overline{D}\right)=\nabla\cdot\frac{\partial\overline{D}}{\partial t}$$

$$\overline{J}_d=\frac{\partial\overline{D}}{\partial t} \longleftarrow \text{displacement current density}$$

$$\boxed{\nabla\times\overline{H}=\overline{J}+\frac{\partial\overline{D}}{\partial t}}$$

$$\int_S\nabla x\overline{H}\cdot\overline{ds}=\int_S\left(\overline{J}+\frac{\partial\overline{D}}{\partial t}\right)\cdot\overline{ds} \longrightarrow \boxed{\oint_L\overline{H}\cdot\overline{dl}=I+\int_S\frac{\partial\overline{D}}{\partial t}\cdot\overline{ds}}$$

Stokes' theorem

Displacement Current continued

A typical example of displacement current is the current through a capacitor when an alternating voltage source is applied to its plates. The following example illustrates the need for the displacement current.

Using an <u>unmodified</u> form of Ampere's law

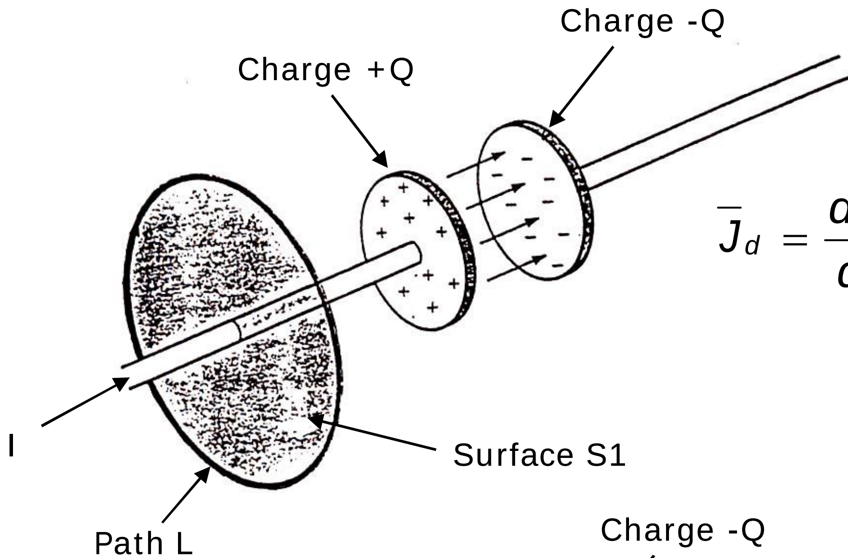
$$\begin{cases} \oint_L \overline{H} \cdot \overline{dl} = \int_{S_1} \overline{J} \cdot \overline{dS} = I_{enc} = I = \dfrac{dQ}{dt} \\ \oint_L \overline{H} \cdot \overline{dl} = \int_{S_2} \overline{J} \cdot \overline{dS} = I_{enc} = 0 \end{cases}$$

$\boxed{\overline{J}_d = 0}$

$\overline{J}_d$

(no conduction current flows through S_2 ($\overline{J} = 0$))

Displacement Current continued



The total current density is $\overline{J} + \overline{J}_d$. In the first equation $\overline{J}_d = 0$ so it remains valid. In the second equation $\overline{J} = 0$ so that

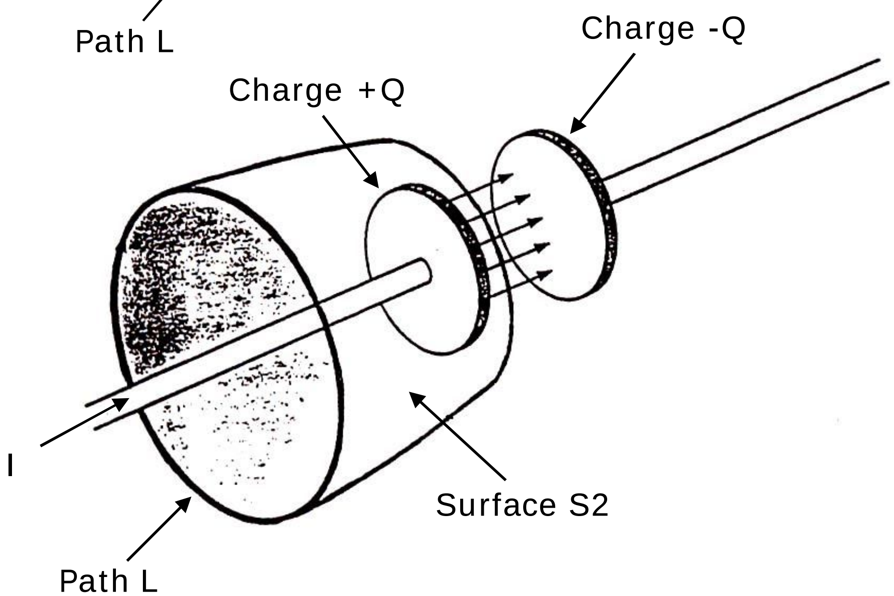
$$\overline{J}_d = \frac{d\overline{D}}{dt}$$

$$\oint_L \overline{H} \cdot d\overline{l} = \int_{S_2} \overline{J}_d \cdot d\overline{S} = \frac{d}{dt} \overbrace{\int_{S_2} \overline{D} \cdot d\overline{S}}^{Q} \leftarrow \boxed{\overline{J} = 0}$$

$$= \frac{dQ}{dt} = I$$

$$\oint_{S_1+S_2} \overline{D} \cdot d\overline{S} = Q$$

$$\int_{S_1} \overline{D} \cdot d\overline{S} = 0$$



So we obtain the same current for either surface though it is conduction current in S_1 and displacement current in S_2 .

Faraday's Law

Faraday discovered experimentally that a current was induced in a conducting loop when the magnetic flux linking the loop changed. In differential (or point) form this experimental fact is described by the following equation

$$\boxed{\nabla x \overline{E} = -\frac{\partial \overline{B}}{\partial t}}$$

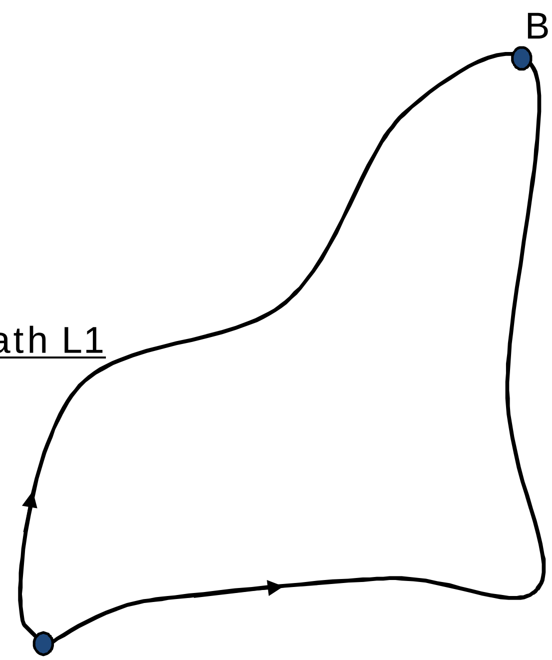
Taking the surface integral of both sides over an open surface and applying Stokes' theorem, we obtain

Integral form $\longrightarrow$ $\oint_L \overline{E} \cdot \overline{dl} = -\frac{\partial}{\partial t}\int_S \overline{B} \cdot \overline{dS} = -\frac{\partial \psi}{\partial t}$

where ψ is the magnetic flux through the surface S.

Faraday's Law continued

Time-varying electric field is not conservative.



The effect of electromagnetic induction. When time-varying magnetic fields are present, the value of the line integral of $\overline{E}$ from A to B may depend on the path one chooses.

Suppose that there is only one unique voltage $V_{AB} = V_A - V_B$. Then

$$V_{AB} = \oint_{L_1} \overline{E} \cdot d\overline{l} = \oint_{L_2} \overline{E} \cdot d\overline{l}$$

However,

$$\oint_L \overline{E} \cdot \overline{dl} = \underbrace{\int_{L_1} \overline{E} \cdot \overline{dl}}_{V_{AB}} - \underbrace{\int_{L_2} \overline{E} \cdot \overline{dl}}_{V_{AB}} = -\frac{\partial \psi}{\partial t}$$

$$V_{AB} - V_{AB} \neq 0$$

$\partial \psi / \partial t = 0$

Thus V_{AB} can be unambiguously defined only if .(in practice, if $\lambda >>$ than the dimensions of system in question)

Faraday's Law continued

According to Faraday's law, a time-varying magnetic flux through a loop of wire results in a voltage across the loop terminals:

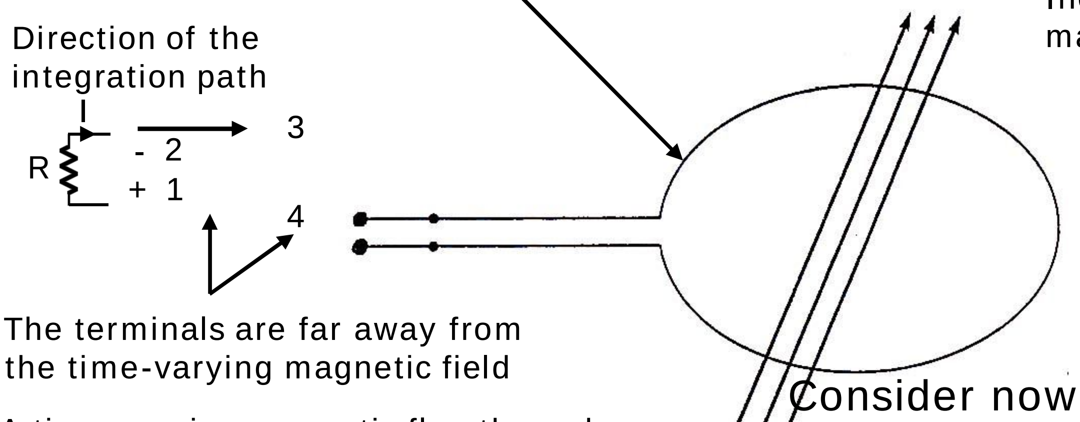
$$V = -\frac{\partial \psi}{\partial t}$$

The negative sign shows that the induced voltage acts in such a way as to <u>oppose</u> the change of flux producing it (<u>Lenz's law</u>).

$$\oint_L \overline{E} \cdot \overline{dl} = \int_1^2 \overline{E} \cdot \overline{dl} + \int_2^1 \overline{E} \cdot \overline{dl} = -\frac{\partial \psi}{\partial t}$$

Induced magnetic field $\overline{B}_{ind}(t)$ (when circuit is closed)

Direction of the integration path



Increasing time-varying magnetic field $B(t)$

$$\int_1^2 \overline{E} \cdot \overline{dl} = V_{12} = -\frac{\partial \psi}{\partial t}$$

$$V_{12} = -N\frac{\partial \psi}{\partial t}$$

If N-turn coil instead of single loop

The terminals are far away from the time-varying magnetic field

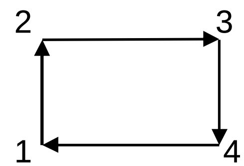
A time-varying magnetic flux through a loop wire results in the appearance of a voltage across its terminals.

No contribution from 2-3 and 4-1 because the wire is a perfect conductor

Consider now terminals 3 and 4, and 1 and 2

$$\int_1^2 \overline{E} \cdot d\bar{l} - \int_4^3 \overline{E} \cdot d\bar{l} = 0$$

$$V_{12} = V_{43}$$



Integration path

Faraday's Law continued

Boundary Condition on Tangential Electric Field

Using Faraday's law, $\oint_L \overline{E} \cdot \overline{dl} = -\partial\psi / \partial t$, we can obtain boundary condition on the tangential component of $\overline{E}$ at a dielectric boundary.

$$\oint_L \overline{E} \cdot \overline{dl} = -\frac{\partial\psi}{\partial t}$$

(unless $B \to \infty$)

$$\int_{L_1} \overline{E} \cdot \overline{dl} + \int_{L_3} \overline{E} \cdot \overline{dl} = 0$$

$L_2 \to 0 \qquad L_4 \to 0 \qquad \psi \to 0$

$$E_{\tau 1} l - E_{\tau 2} l = 0$$

$\boxed{E_{\tau 1} = E_{\tau 2}}$ - continuous at the boundary

(For the normal component: $D_{N1} = D_{N2}$)
at a dielectric boundary

General Forms of Maxwell's Equations

	Differential	**Integral**	**Remarks**
1	$\nabla \cdot \overline{D} = \rho_V$	$\oint_S \overline{D} \cdot d\overline{s} = \int_V \rho_V \, dv$	Gauss' Law
2	$\nabla \cdot \overline{B} = 0$	$\oint_S \overline{B} \cdot d\overline{s} = 0$	Nonexistence of isolated magnetic charge
3	$\nabla x \overline{E} = -\frac{\partial \overline{B}}{\partial t}$	$\oint_L \overline{E} \cdot d\overline{l} = -\frac{\partial}{\partial t} \int_s \overline{B} \cdot d\overline{s}$	Faraday's Law
4	$\nabla x \overline{H} = \overline{J} + \frac{\partial \overline{D}}{\partial t}$	$\oint_L \overline{H} \cdot d\overline{l} = \int_s \left(\overline{J} + \frac{\partial \overline{D}}{\partial t} \right) \cdot d\overline{s}$	Ampere's circuital law

In 1 and 2, S is a closed surface enclosing the volume V

In 2 and 3, L is a closed path that bounds the surface S

Sinusoidal Fields

In electromagnetics, information is usually transmitted by imposing amplitude, frequency, or phase modulation on a sinusoidal carrier. Sinusoidal (or time-harmonic) analysis can be extended to most waveforms by Fourier and Laplace transform techniques.

Sinusoids are easily expressed in phasors, which are more convenient to work with. Let us consider the "curl H" equation.

$$\nabla\times\overline{H}=\overline{J}+\frac{\partial\overline{D}}{\partial t} \qquad\qquad \overline{H}=f\left(x,y,z,t\right)$$

Its phasor representation is

$$\nabla\times\underline{\overline{H}}=\underline{\overline{J}}+j\omega\underline{\overline{D}}=\underline{\overline{J}}+j\omega\varepsilon\underline{\overline{E}}$$

$\underline{\overline{H}}$ is a vector function of position, but it is independent of time. The three scalar components of $\overline{H}$ are complex numbers; that is, if

$$\overline{H}(x,y,z,t)=f_1(x,y,z)\cos(\omega t+\phi_1)\overline{e}_x+f_2(x,y,z)\cos(\omega t+\phi_2)\overline{e}_y$$

then

$$\underline{\overline{H}}(x,y,z)=f_1(x,y,z)e^{j\phi_1}\overline{e}_x+f_2(x,y,z)e^{j\phi_2}\overline{e}_y$$

Sinusoidal Fields continued

Time-Harmonic Maxwell's Equations Assuming Time Factor $e^{j\omega t}$.

Point Form

$$\nabla \cdot \underline{\overline{D}} = \underline{\rho_V}$$

$$\nabla \cdot \underline{\overline{B}} = 0$$

$$\nabla \times \underline{\overline{E}} = -j\omega \underline{\overline{B}}$$

$$\nabla \times \underline{\overline{H}} = \underline{\overline{J}} + j\omega \underline{\overline{D}}$$

Integral Form

$$\oint \underline{\overline{D}} \cdot d\overline{s} = \int \underline{\rho_V}\, dv$$

$$\oint \underline{\overline{B}} \cdot d\overline{s} = 0$$

$$\oint \underline{\overline{E}} \cdot d\overline{l} = -j\omega \int \underline{\overline{B}} \cdot d\overline{s}$$

$$\oint \underline{\overline{H}} \cdot d\overline{l} = \int \left(\underline{\overline{J}} + j\omega \underline{\overline{D}} \right) \cdot d\overline{s}$$

$$\overline{H}(x,y,z,t) = \mathrm{Re}\left[\underline{\overline{H}}(x,y,z) e^{j\omega t} \right]$$

Poynting's Theorem

$$-\int_V (\mathbf{E}\cdot\mathbf{J}_0)\,dv = \int_V \sigma E^2\,dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\,\varepsilon E^2\right)dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\,\mu H^2\right)dv + \oint_S \mathbf{P}\cdot d\mathbf{S}$$

Source power density

Power dissipation density

Electric stored energy density

Magnetic stored energy density

Interpretation of Poynting's Theorem

Poynting's Theorem for the material medium says that the power delivered to the volume V by the current source $\mathbf{J}_0$ is accounted for by the power dissipated in the volume due to the conduction current in the medium, plus the time rates of increase of the energies stored in the electric and magnetic fields, plus another term, which we must interpret as the power carried by the electromagnetic field out of the volume V, for conservation of energy to be satisfied.

$\mathbf{P} = \mathbf{E} \times \mathbf{H} =$ Power flow density $\left(\mathrm{W/m^2}\right)$ associated with the electromagnetic field

$p_d = \sigma E^2 =$ Power dissipation density due to conduction current flow in the material

$$w_e = \frac{1}{2}\varepsilon E^2 = \text{ Energy density } \left(\mathbf{J}/\mathrm{m}^3\right) \text{stored in the electric field}$$

$$w_m = \frac{1}{2}\mu H^2 = \text{ Energy density } \left(\mathbf{J}/\mathrm{m}^3\right) \text{stored in the magnetic field}$$

UNIT II

TRANSMISSION LINES

Transmission Line Concepts

- The transmission line is divided into small units where the circuit elements can be lumped.
- Assuming the resistance of the lines is zero, then the transmission line can be modeled as an LC ladder network with inductors in the series arms and the capacitors in the shunt arms.
- The value of inductance and capacitance of each part determines the velocity of propagation of energy down the line.
- Time taken for a wave to travel one unit length is equal to

$$T(s) = (LC)^{0.5}$$

- Velocity of the wave is equal to

$$v\ (m/s) = 1/T$$

- Impedance at any point is equal to

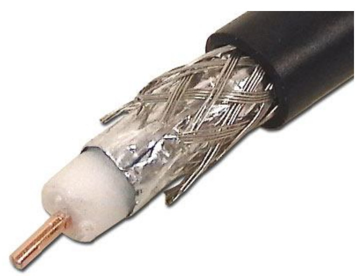
$$Z = V\ (\text{at any point})/I\ (\text{at any point})$$

$$Z = (L/C)^{0.5}$$

Transmission Line

Properties

- Has two conductors running parallel
- Can propagate a signal at any frequency (in theory)
- Becomes lossy at high frequency
- Can handle low or moderate amounts of power
- Does not have signal distortion, unless there is loss
- May or may not be immune to interference
- Does not have E_z or H_z components of the fields (TEM$_z$)



Coaxial cable (coax)

Twin lead

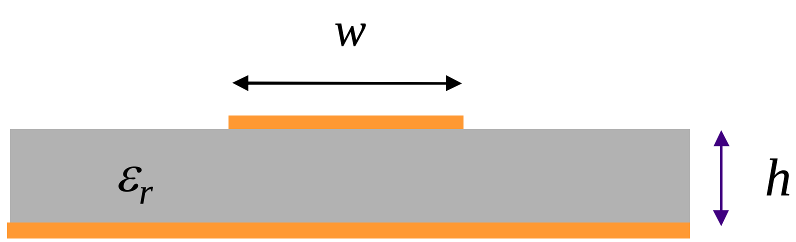
Transmission Line (cont.)

CAT 5 cable
(twisted pair)

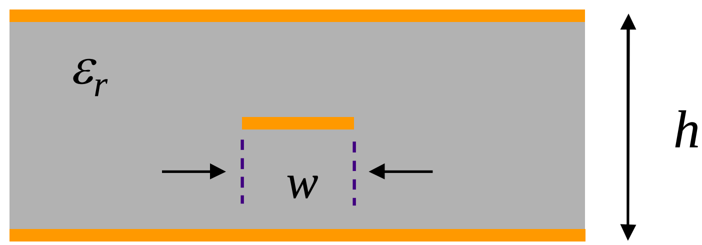
The two wires of the transmission line are twisted to reduce interference and radiation from discontinuities.

Transmission Line (cont.)

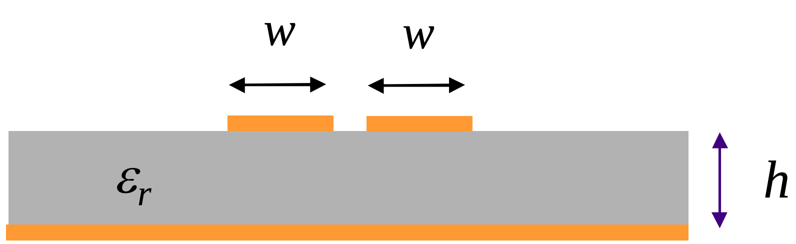
Transmission lines commonly met on printed-circuit boards



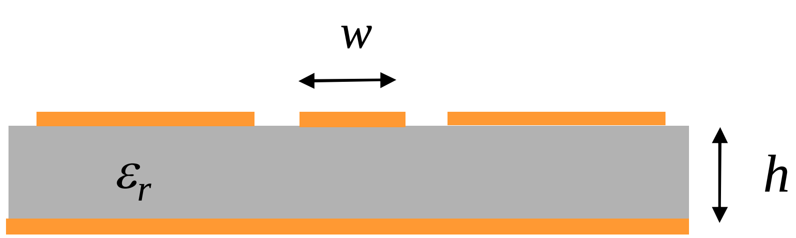
Microstrip



Stripline



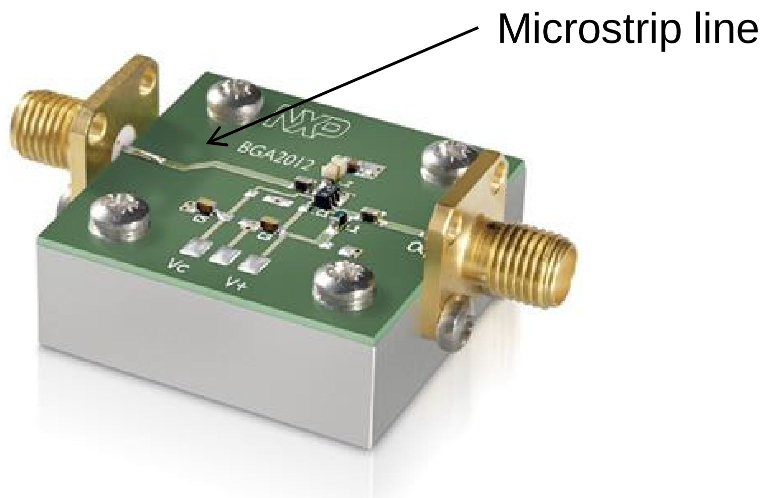
Coplanar strips



Coplanar waveguide (CPW)

Transmission Line (cont.)

Transmission lines are commonly met on printed-circuit boards.

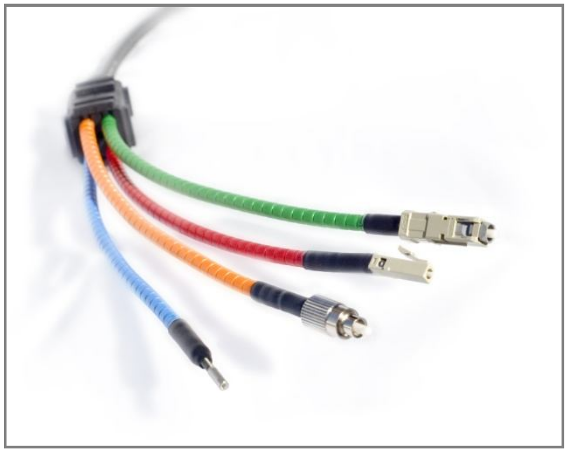


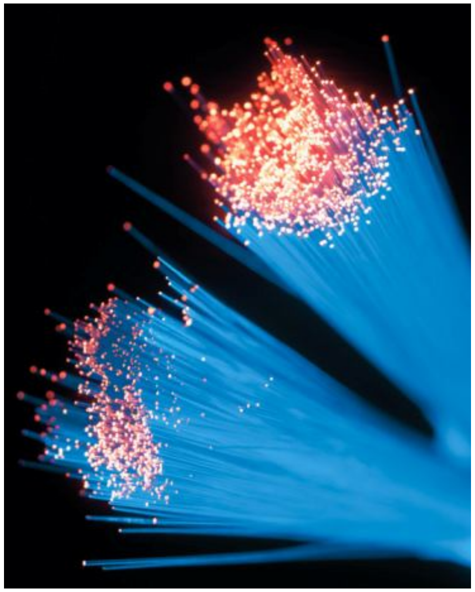
A microwave integrated circuit

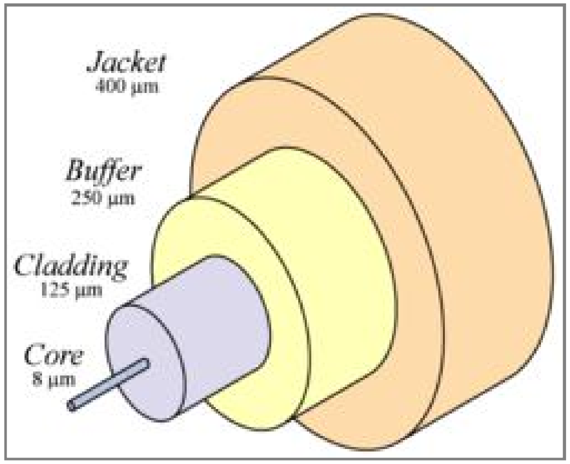
Fiber-Optic Guide

Properties

- Uses a dielectric rod
- Can propagate a signal at any frequency (in theory)
- Can be made very low loss
- Has minimal signal distortion
- Very immune to interference
- Not suitable for high power
- Has both E_z and H_z components of the fields







Fiber-Optic Guide (cont.)

Two types of fiber-optic guides:

1) Single-mode fiber

Carries a single mode, as with the mode on a transmission line or waveguide. Requires the fiber diameter to be small relative to a wavelength.

2) Multi-mode fiber

Has a fiber diameter that is large relative to a wavelength. It operates on the principle of total internal reflection (critical angle effect).

TEM Transmission Line (cont.)

To combine these, take the derivative of the first one with respect to z:

$$
\begin{aligned}
\frac{\partial^2 v}{\partial z^2} &= -R\frac{\partial i}{\partial z} - L\frac{\partial}{\partial z}\left(\frac{\partial i}{\partial t}\right) \\
&= -R\frac{\partial i}{\partial z} - L\frac{\partial}{\partial t}\left(\frac{\partial i}{\partial z}\right) \\
&= -R\left[-Gv - C\frac{\partial v}{\partial t}\right] \\
&\quad - L\left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]
\end{aligned}
$$

Switch the order of the derivatives.

TEM Transmission Line (cont.)

$$\frac{\partial^2 v}{\partial z^2} = -R\left[-Gv - C\frac{\partial v}{\partial t}\right] - L\left[-G\frac{\partial v}{\partial t} - C\frac{\partial^2 v}{\partial t^2}\right]$$

Hence, we have:

$$\frac{\partial^2 v}{\partial z^2} - \left(RG\right)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

The same equation also holds for i.

TEM Transmission Line (cont.)

Time-Harmonic Waves:

$$\frac{\partial^2 v}{\partial z^2} - (RG)v - (RC + LG)\frac{\partial v}{\partial t} - LC\left(\frac{\partial^2 v}{\partial t^2}\right) = 0$$

$$\frac{d^2 V}{dz^2} - (RG)V - (RC + LG)j\omega V - LC(-\omega^2)V = 0$$

TEM Transmission Line (cont.)

$$\frac{d^2V}{dz^2} = \left(RG\right)V + j\omega(RC + LG)V - \left(\omega^2 LC\right)V$$

Note that

$$RG + j\omega(RC + LG) - \omega^2 LC = (R + j\omega L)(G + j\omega C)$$

$$Z = R + j\omega L$$ = series impedance/length

$$Y = G + j\omega C$$ = parallel admittance/length

Then we can write:

$$\frac{d^2V}{dz^2} = (ZY)V$$

TEM Transmission Line (cont.)

Let $\boxed{\gamma^2 = ZY}$ Then $\boxed{\dfrac{d^2V}{dz^2} = (\gamma^2)V}$

Solution: $\boxed{V(z) = Ae^{-\gamma z} + Be^{+\gamma z}}$

γ is called the "propagation constant."

Convention:

$$\boxed{\begin{aligned}\gamma &= \left[(R + j\omega L)(G + j\omega C)\right]^{1/2} \\ &= \text{principal square root}\end{aligned}}$$

$$\boxed{\sqrt{z} = \sqrt{|z|}\, e^{j\theta/2}}$$

$$\boxed{-\pi < \theta < \pi}$$

$$\boxed{\gamma = \alpha + j\beta}$$

$$\boxed{\alpha \geq 0,\ \beta \geq 0}$$

α = attenuation contant

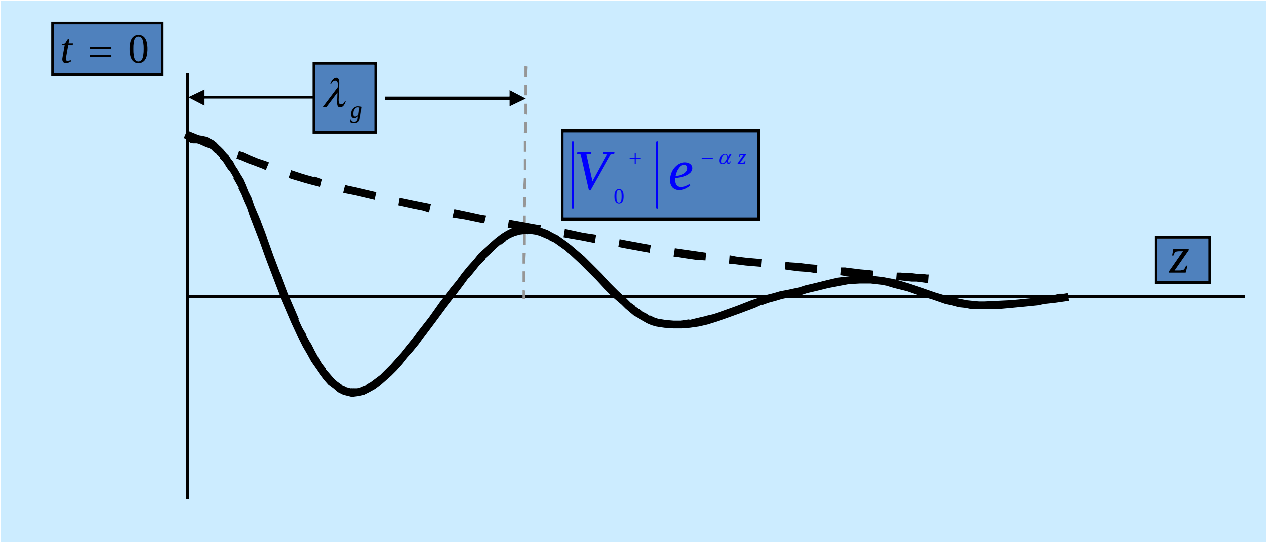
β = phase constant

TEM Transmission Line (cont.)

Forward travelling wave (a wave traveling in the positive z direction):

$$V^+(z) = V_0^+ e^{-\gamma z} = V_0^+ e^{-\alpha z} e^{-j\beta z}$$

$$\begin{aligned} v^+(z,t) &= \mathrm{Re}\left\{\left(V_0^+ e^{-\alpha z} e^{-j\beta z}\right) e^{j\omega t}\right\} \\ &= \mathrm{Re}\left\{\left(\left|V_0^+\right| e^{j\phi} e^{-\alpha z} e^{-j\beta z}\right) e^{j\omega t}\right\} \\ &= \left|V_0^+\right| e^{-\alpha z} \cos\left(\omega t - \beta z + \phi\right) \end{aligned}$$



The wave "repeats" when:

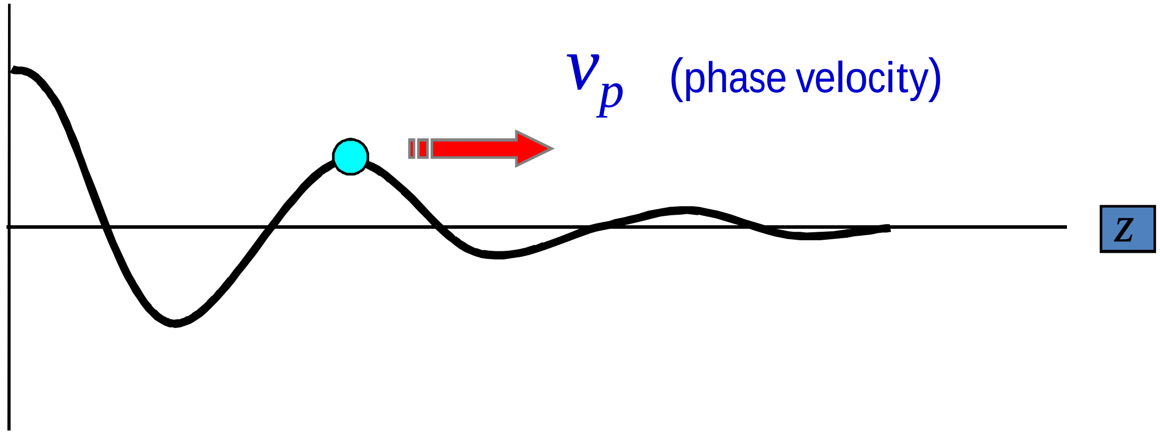
$$\beta \lambda_g = 2\pi$$

Hence:

$$\beta = \frac{2\pi}{\lambda_g}$$

Phase Velocity

Track the velocity of a fixed point on the wave (a point of constant phase), e.g., the crest.



$$v^{+}(z,t) = \left|V_0^{+}\right| e^{-\alpha z} \cos(\omega t - \beta z + \phi)$$

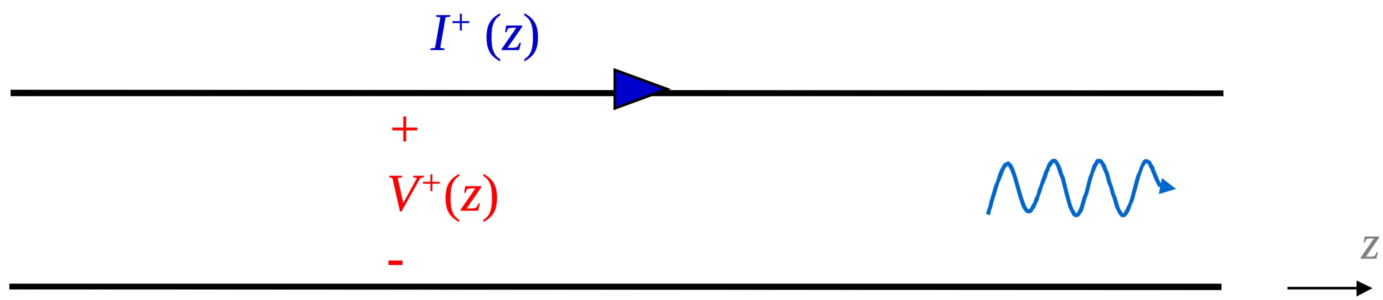
Phase Velocity (cont.)

Set
$$\omega t - \beta z = \text{constant}$$
$$\omega - \beta \frac{dz}{dt} = 0$$
$$\frac{dz}{dt} = \frac{\omega}{\beta}$$

Hence
$$\boxed{v_p = \frac{\omega}{\beta}}$$

In expanded form:
$$v_p = \frac{\omega}{\mathrm{Im}\left\{\left[(R + j\omega L)(G + j\omega C)\right]^{1/2}\right\}}$$

Characteristic Impedance Z_0



A wave is traveling in the positive *z* direction.

$$Z_0 \equiv \frac{V^+(z)}{I^+(z)}$$

$$V^+(z) = V_0^+ e^{-\gamma z}$$

$$I^+(z) = I_0^+ e^{-\gamma z}$$

so

$$Z_0 = \frac{V_0^+}{I_0^+}$$

(Z_0 is a number, not a function of z.)

Characteristic Impedance Z_0 (cont.)

Use Telegrapher's Equation:

$$\boxed{\frac{\partial v}{\partial z} = -Ri - L\frac{\partial i}{\partial t}}$$

so

$$\boxed{\begin{aligned}\frac{dV}{dz} &= -RI - j\omega LI \\ &= -ZI\end{aligned}}$$

Hence

$$\boxed{-\gamma V_0^{+} e^{-\gamma z} = -ZI_0^{+} e^{-\gamma z}}$$

Characteristic Impedance Z_0 (cont.)

From this we have:

$$Z_0 = \frac{V_0^+}{I_0^+} = \frac{Z}{\gamma} = \left(\frac{Z}{Y}\right)^{1/2}$$

Using

$$Z = R + j\omega L$$

$$Y = G + j\omega C$$

We have

$$Z_0 = \left(\frac{R + j\omega L}{G + j\omega C}\right)^{1/2}$$

Note: The principal branch of the square root is chosen, so that Re $(Z_0) > 0$.

General Case (Waves in Both Directions)

$$V(z) = V_0^+ e^{+\gamma z} + V_0^- e^{+\gamma z}$$

$$= \left|V_0^+\right| e^{j\phi^+} e^{+\alpha z} e^{-j\beta z} + \left|V_0^-\right| e^{j\phi^-} e^{+\alpha z} e^{+j\beta z}$$

Note:

wave in $+z$ direction

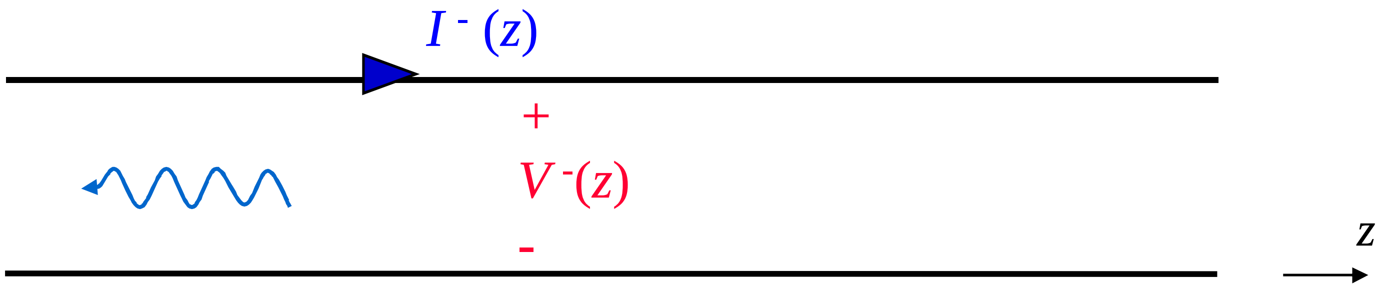
wave in $-z$ direction

$$v(z,t) = \mathrm{Re}\left\{V(z)e^{j\omega t}\right\}$$

$$= \left|V_0^+\right| e^{-\alpha z} \cos\left(\omega t - \beta z + \phi^+\right)$$

$$+ \left|V_0^-\right| e^{+\alpha z} \cos\left(\omega t + \beta z + \phi^-\right)$$

Backward-Travelling Wave



A wave is traveling in the negative z direction.

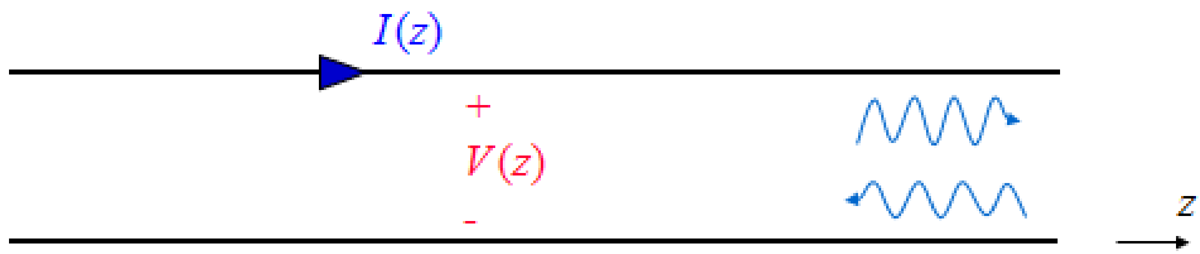
$$\boxed{\frac{V^{-}(z)}{-I^{-}(z)} = Z_0}$$

so

$$\boxed{\frac{V^{-}(z)}{I^{-}(z)} = -Z_0}$$

Note: The reference directions for voltage and current are the same as for the forward wave.

GENERAL



Most general case:

A general superposition of forward and backward traveling waves:

$$V(z) = V_0^{+} e^{-\gamma z} + V_0^{-} e^{+\gamma z}$$

$$I(z) = \frac{1}{Z_0}\left[V_0^{+} e^{-\gamma z} - V_0^{-} e^{+\gamma z}\right]$$

Note: The reference directions for voltage and current are the same for forward and backward waves.

LOSSLESS PROPAGATION

$$R = 0,\ G = 0$$

$$\gamma = \alpha + j\beta = [(\cancel{R} + j\omega L)(\cancel{G} + j\omega C)]^{1/2}$$
$$= j\omega\sqrt{LC}$$

so

$$\alpha = 0$$
$$\beta = \omega\sqrt{LC}$$

$$v_p = \frac{\omega}{\beta}$$

$$Z_0 = \left(\frac{\cancel{R} + j\omega L}{\cancel{G} + j\omega C}\right)^{1/2}$$

$$Z_0 = \sqrt{\frac{L}{C}}$$

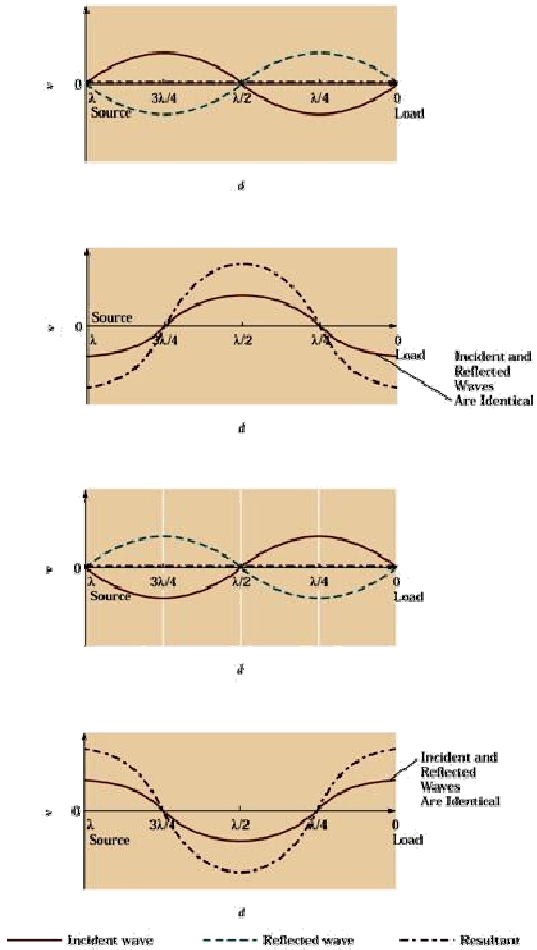
(real and indep. of freq.)

$$v_p = \frac{1}{\sqrt{LC}}$$

(indep. of freq.)

Standing Waves

- **The interaction of incident and reflected waves in a transmission line results in *standing waves***
- **When a reflected wave is present but has lower amplitude than the incident, there will be no point on the line where the voltage or current remains zero over the whole cycle**



Voltage Standing Wave Ratio

- When a transmission line is terminated with an impedance, Z_L , that is not equal to the characteristic impedance of the transmission line, Z_0 , not all of the incident power is absorbed by the termination. Part of the power is reflected back , so that phase addition and subtraction of the incident and reflected waves creates a voltage standing wave pattern on the transmission line. The ratio of the maximum to minimum voltage is known as the Voltage Standing Wave Ratio (VSWR) and successive maxima and minima are spaced by 180 degrees.

$$VSWR = \frac{E_{max}}{E_{min}} = \frac{E_i + E_r}{E_i - E_r}$$

where

E_{max} = maximum voltage on the standing wave

E_{min} = minimum voltage on the standing wave

E_i = incident voltage wave amplitude

E_r = reflected voltage wave amplitude

Reflection Co-efficient

The reflection coefficient, ρ, is defined as E_r/E_i and in general, the termination is complex in value, so that ρ will be a complex number.

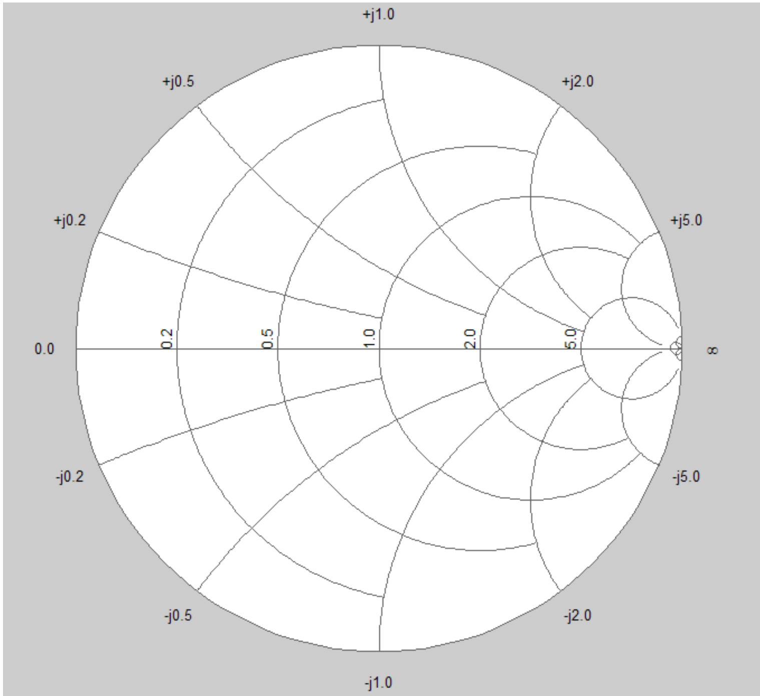
Additionally we define: $\Gamma = \dfrac{Z_L - Z_O}{Z_L + Z_O}$ The refection coefficient, ρ, is the absolute value of the magnitude of Γ.

If the equation for VSWR is solved for the reflection coefficient, it is found that:

$$\textit{Reflection Coefficient} = \rho = |\Gamma| = \frac{VSWR-1}{VSWR+1} \quad \text{Consequently,} \quad VSWR = \frac{1+\rho}{1-\rho}$$

Smith Chart

- Smith Chart is a graphical plot of the normalized resistance and reactance functions in the complex reflection-coefficient plane. It is a graph showing both the normalized impedance and the reflection coefficient. Smith chart is convenient for transmission line and circuit calculations. It is also a useful tool in impedance matching circuit design.



Smith Chart (contd..)

Recall that:

$$\Gamma(\ell) = \frac{Z(\ell) - Z_0}{Z(\ell) + Z_0}$$

$$Z(\ell) = Z_0 \frac{1 + \Gamma(\ell)}{1 - \Gamma(\ell)}$$

Now normalize the impedance $Z(\ell)$ by Z_0.

$$z = \frac{Z(\ell)}{Z_0}$$

In terms of the normalized impedance z (drop the ℓ dependence), we can write:

$$\Gamma = \frac{z-1}{z+1} = \Gamma_{re} + j\Gamma_{im}$$

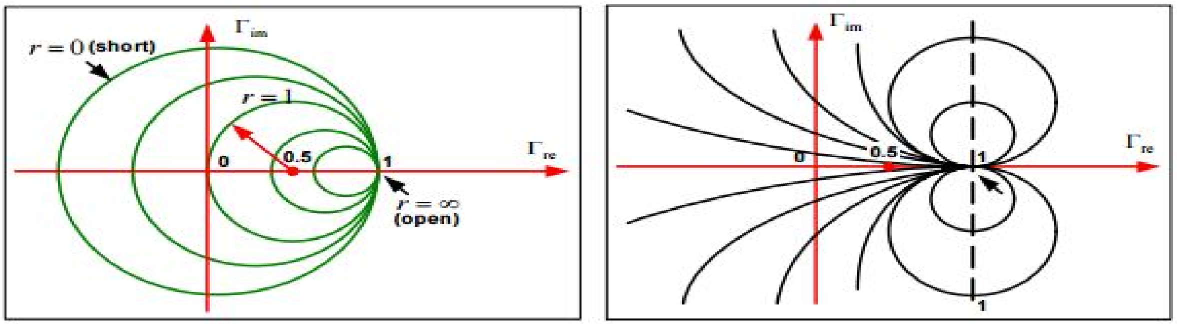
$$z = r + jx = \frac{1+\Gamma}{1-\Gamma} = \frac{(1+\Gamma_{re}) + j\Gamma_{im}}{(1-\Gamma_{re}) - j\Gamma_{im}}$$

From the last equation, we have

$$r = \frac{1 - \Gamma_{re}{}^2 - \Gamma_{im}{}^2}{(1-\Gamma_{re})^2 + \Gamma_{im}{}^2} \qquad x = \frac{2\Gamma_{im}}{(1-\Gamma_{re})^2 + \Gamma_{im}{}^2}$$

Smith Chart (contd..)

The last two equations of r and x define two families of circles in the complex plane of reflection coefficient Γ.



The Smith char is the superposition of these two families of circles together in the complex plane of reflection coefficient Γ.

- Voltage Maxima and Minima in Smith Chart Voltage maxima occur when the angle of the reflection coefficient (Γ (ℓ)) θ = -2 n π (n = 0, 1, 2, …). This corresponds to the right-most point in the Smith chart. Voltage minima occur when the angle of the reflection coefficient (Γ (ℓ)) θ = -2(n+1) π (n = 0, 1, 2, …). This corresponds to the left-most point in the Smith chart.

Impedance Matching

Reasons for impedance matching:

1. Maximize power transfer to the load

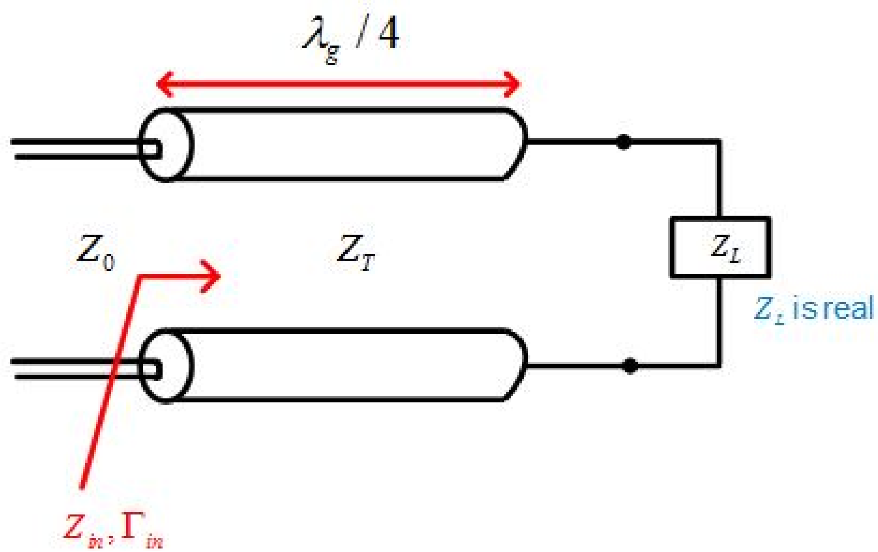
2. The input impedance remains constant at the value Z_0. Therefore, the input impedance is independent of the length of transmission line.
3. VSWR = 1. Therefore there are no voltage peaks on the transmission line.

Meaning of impedance matching: Impedance matching is to eliminate the reflected voltage or current on a transmission line.

Two matching techniques:

1. Quarter-wave transformer
2. Single-stub matching network

Quarter Wave Transformer



$@\ f = f_0$

$$\beta \ell = \frac{2\pi}{\lambda_g} \frac{\lambda_g}{4} = \frac{\pi}{2}$$

$\Gamma_{in} = 0$ when $Z_T = \sqrt{Z_0 Z_L}$

$$Z_{in} = \frac{Z_T^2}{Z_L}$$

Only true at f_0 where $\ell = \lambda_g / 4$

Note: If Z_L is not real, we can always add a reactive load in series or parallel to make it real (or add a length of transmission line between the load and the transformer to get a real impedance).

Quarter Wave Transformer (contd.)

At a general frequency:

$$Z_{in} = Z_T \left(\frac{Z_L + jZ_T \tan \beta_T \ell}{Z_T + jZ_L \tan \beta_T \ell} \right)$$

$$= Z_T \left(\frac{Z_L + jZ_T t}{Z_T + jZ_L t} \right) \qquad t \equiv \tan \beta_T \ell$$

$$\Gamma = \frac{Z_{in} - Z_0}{Z_{in} + Z_0} = \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}}$$

where we have used $Z_T = \sqrt{Z_0 Z_L}$

Quarter Wave Transformer (contd.)

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0 + j2t\sqrt{Z_0 Z_L}}$$

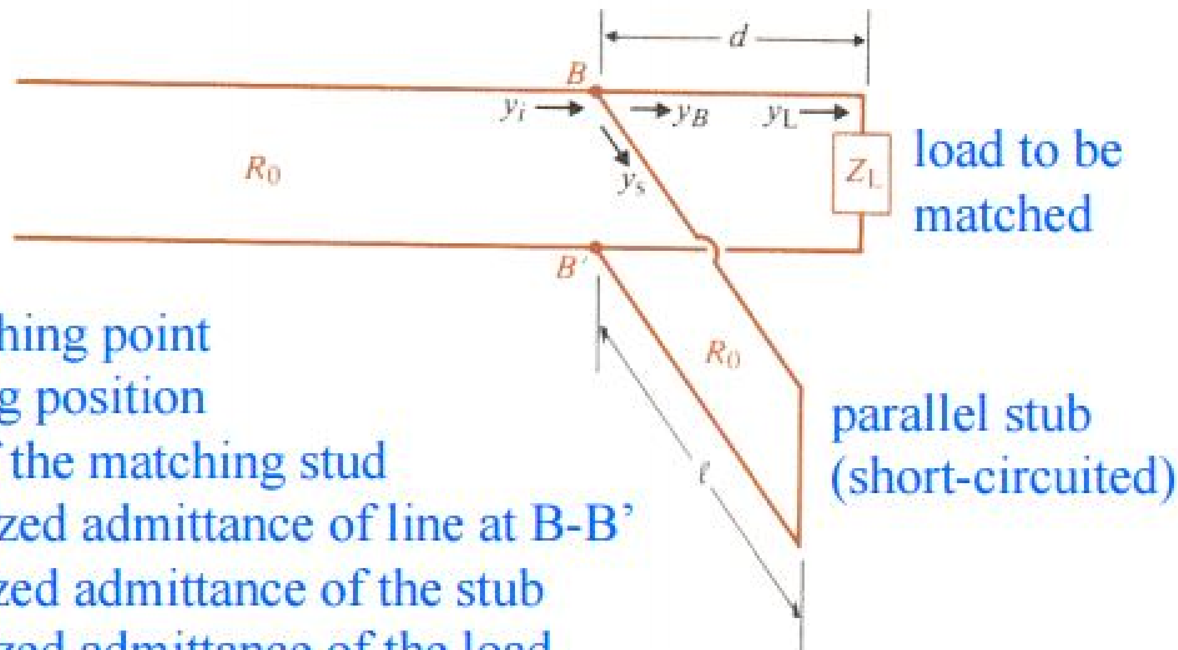
$$|\Gamma| = \frac{1}{\sqrt{1 + \left(\frac{4 Z_0 Z_L}{(Z_L - Z_0)^2} \sec^2 \beta_T \ell \right)}}$$

where we used $1 + t^2 = 1 + \tan^2 \beta_T l = \sec^2 \beta_T l$

Single stub Matching

What is a stub?

A stub is a short section of transmission line (shorted or open at one end) whose input impedance can be changed by varying its length.

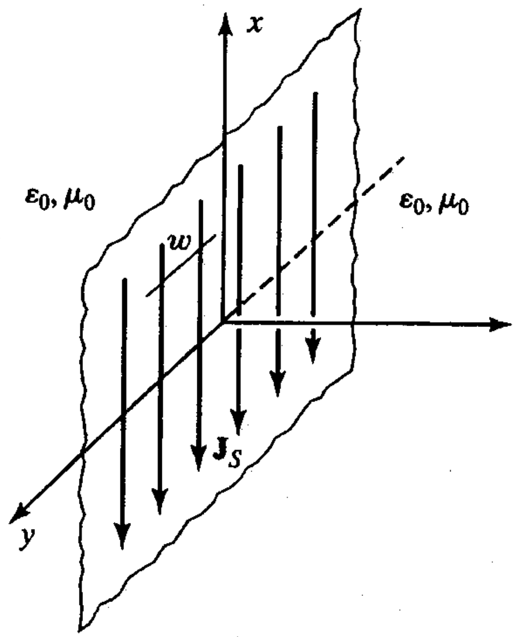


B-B ' = matching point
d = matching position
ℓ = length of the matching stud
y_B = normalized admittance of line at B-B'
y_s = normalized admittance of the stub
y_L = normalized admittance of the load

UNIT-III

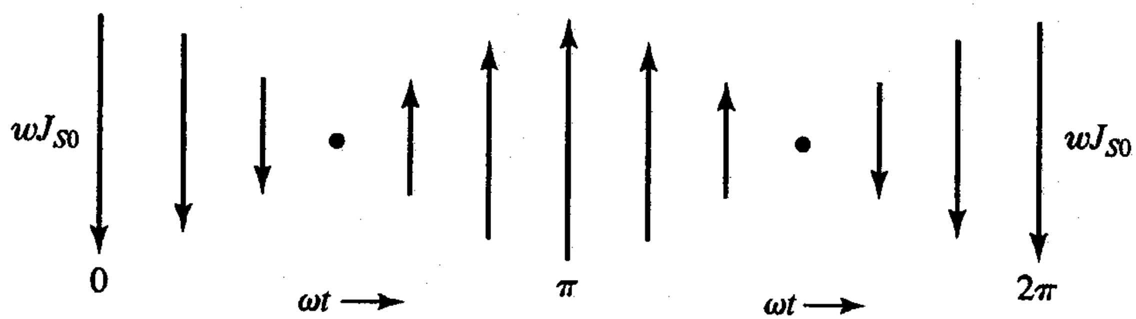
THE UNIFORM PLANE WAVE

Infinite Plane Current Sheet Source



$$\mathbf{J}_S = -J_S(t)\,\mathbf{a}_x$$

for z = 0



Example: $\mathbf{J}_S(t) = J_{S0} \cos \omega t \; \mathbf{a}_x$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

For a current distribution having only an *x*-component of current density that varies only with *z*,

$$\begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$-\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \qquad -\frac{\partial H_y}{\partial z} = J_x + \frac{\partial D_x}{\partial t}$$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad \frac{\partial H_x}{\partial z} = \frac{\partial D_y}{\partial t}$$

$$0 = -\frac{\partial B_z}{\partial t} \qquad 0 = \frac{\partial D_z}{\partial t}$$

The only relevant equations are:

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \qquad -\frac{\partial H_y}{\partial z} = J_x + \frac{\partial \mathbf{D}_x}{\partial t}$$

Thus, $\mathbf{E} = E_x(z, t)\mathbf{a}_x \qquad \mathbf{H} = H_y(z, t)\mathbf{a}_y$

In the free space on either side of the sheet, $J_x = 0$

$$\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} = -\mu_0 \frac{\partial H_y}{\partial t} \qquad \frac{\partial H_y}{\partial z} = \frac{\partial D_x}{\partial t} = \varepsilon_0 \frac{\partial E_x}{\partial t}$$

Combining, we get

$$\frac{\partial^2 E_x}{\partial z^2} = -\mu_0 \frac{\partial}{\partial z}\left(\frac{\partial H_y}{\partial t}\right)$$

$$= -\mu_0 \frac{\partial}{\partial t}\left(\frac{\partial H_y}{\partial z}\right)$$

$$= -\mu_0 \frac{\partial}{\partial t}\left(-\varepsilon_0 \frac{\partial E_x}{\partial t}\right)$$

$$\boxed{\frac{\partial^2 E_x}{\partial z^2} = \mu_0 \varepsilon_0 \frac{\partial^2 E_x}{\partial t^2}}$$

Wave Equation

Uniform Plane Wave in Dielectric and Conductors

1. Perfect dielectric: $(\sigma = 0)$

$$\bar{\gamma} = \sqrt{j\omega\mu \cdot j\omega\varepsilon} = j\omega\sqrt{\mu\varepsilon}$$

$$\therefore \alpha = 0 \qquad \text{no attenuation}$$

$$\beta = \omega\sqrt{\mu\varepsilon}$$

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{j\omega\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}, \quad \text{purely real}$$

Behavior same as in free space except that $\varepsilon_0 \rightarrow \varepsilon$ and $\mu_0 \rightarrow \mu$.

2. Imperfect Dielectric: $\left(\sigma \neq 0 \text{ but } \sigma \ll \omega\varepsilon\right)$

$$\bar{\gamma} = \sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)}$$

$$\approx \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}} + j\omega\sqrt{\mu\varepsilon}$$

$$\bar{\eta} \approx \sqrt{\frac{\mu}{\varepsilon}}\left(1 + j\frac{\sigma}{2\omega\varepsilon}\right)$$

Behavior essentially like in a perfect dielectric except for attenuation.

3. Good Conductor: $\left(\sigma \gg \omega\varepsilon\right)$

$$\bar{\gamma} = \sqrt{j\omega\mu\left(\sigma + j\omega\varepsilon\right)}$$

$$\approx \sqrt{\pi f \mu\sigma}\ \left(1 + j\right)$$

$$\therefore \alpha = \beta \approx \sqrt{\pi f \mu\sigma}$$

$$\bar{\eta} = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}} \approx \sqrt{\frac{\pi f \mu}{\sigma}}\left(1 + j\right)$$

$$= \sqrt{\frac{2\pi f \mu}{\sigma}}\ \angle 45^\circ$$

Behavior much different from that in a dielectric.

4. Perfect Conductor: $(\sigma = \infty)$

Idealization of good conductor in the limit

that $\sigma \to \infty$.

$$\alpha \to \infty,\ \bar{\eta} \to 0$$

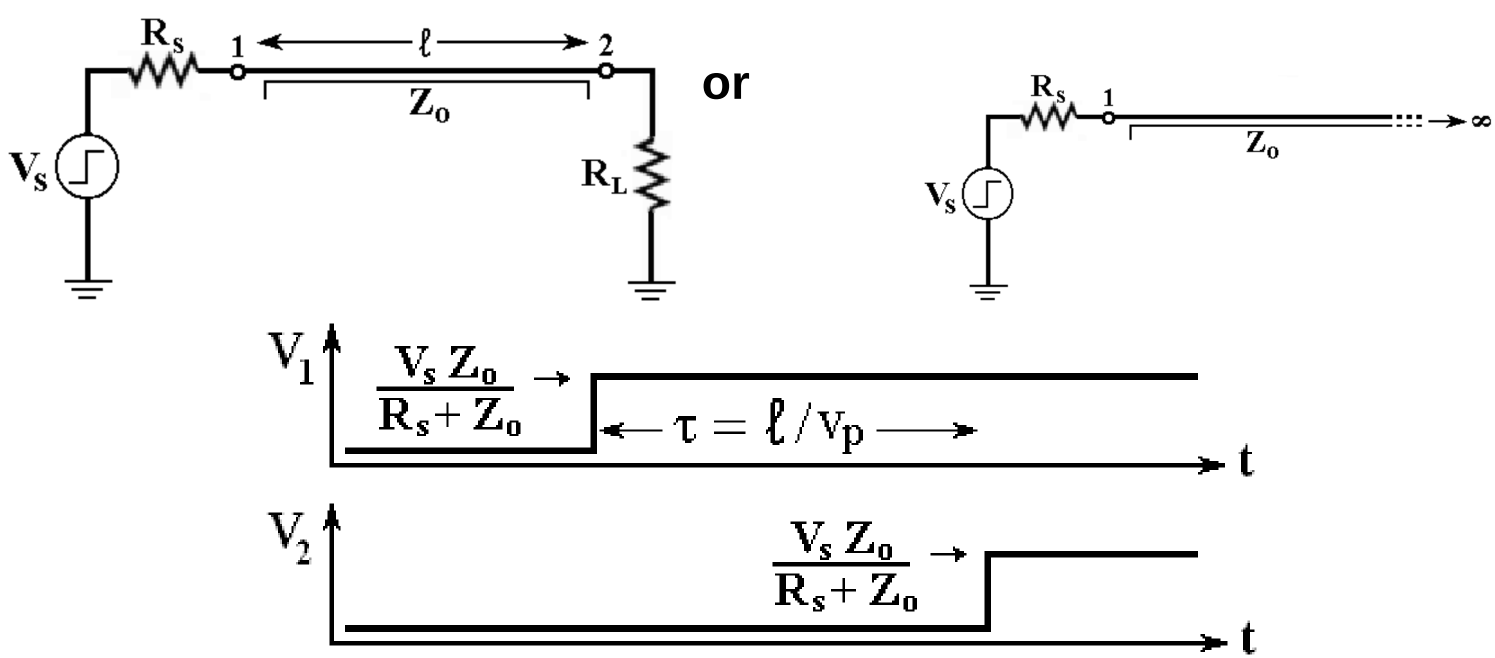
$\therefore$ No waves can penetrate into a perfect conductor.
No time-varying fields inside a perfect conductor.

Transmission line modeling

Travelling waves

–For frequencies above f_1 (and frequency components above f_1), Z_o is real and frequency independent

–For a transmission line with a matched load ($R_L = Z_o$), when 'looking' into a transmission line, the source initially 'sees' a resistive load



Transmission line modeling

Travelling waves example

– Consider the case where the 5-V step voltage generator with a 30-Ω source resistance drives a 50-Ω transmission line with a matched impedance termination.

$$V_s = 5\text{ V},\ R_s = 30\ \Omega,\ Z_o = 50\ \Omega,\ R_L = 50\ \Omega$$

– The wave propagating down the transmission line has a voltage of

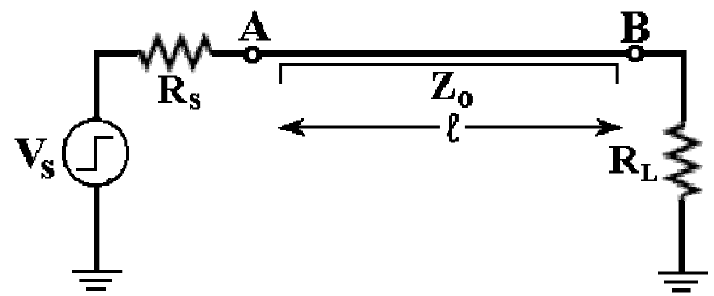
$$5V\,\frac{50}{50+30} = 3.13V$$

– The wave propagating down the transmission line has a current of

$$\frac{3.13\text{ V}}{50\ \Omega} = 63\text{ mA}$$

Transmission line modeling

- Travelling waves
  - Now consider the more general case with impedance mismatches
  - First consider the source end



for $R_s = Z_o$, $V_A = V_s/2$

for $R_s \neq Z_o$, $V_A = V_S \dfrac{Z_o}{R_S + Z_o}$

  - The wave travels down the transmission line and arrives at B at time $t = \ell/v_p$ where the impedance mismatch causes a reflected wave with reflection coefficient, ρ_L

$$\rho_L = \frac{R_L - Z_o}{R_L + Z_o} \quad \rightarrow \text{wave voltage}$$

  - Special cases
    - if $R_L = Z_o$ (matched impedance), then $\rho_L = 0$
    - if $R_L = 0$ (short circuit), then $\rho_L = -1$
    - if $R_L = \infty$ (open circuit), then $\rho_L = +1$

Transmission line modeling

- The reflected signal then travels back down the transmission line and arrives at A at time $t = 2\ell/v_p$ where another impedance mismatch causes a reflected wave with reflection coefficient, ρ_S

$$\rho_S = \frac{R_S - Z_o}{R_S + Z_o} \rightarrow \text{wave voltage} = V_A\, \rho_L\, \rho_S$$

- This reflected signal again travels down the transmission line
- For complex loads, the reflection coefficient is complex
- In general

$$\rho = \frac{Z_L - Z_o}{Z_L + Z_o} \quad \textbf{or} \quad \frac{Z_L}{Z_o} = \frac{1+\rho}{1-\rho}$$

Poynting's Theorem for Power Flow

Consider the quantity $\mathbf{E}\times\mathbf{H}$. Then, from a vector identity,

$$\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)=\mathbf{H}\cdot\left(\nabla\times\mathbf{E}\right)-\mathbf{E}\cdot\left(\nabla\times\mathbf{H}\right)$$

Substituting

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$

$$\nabla\times\mathbf{H}=\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}=\mathbf{J}_0+\frac{\partial\mathbf{D}}{\partial t}$$

where $\mathbf{J}_0$ represents source current density, we have

$$\nabla\cdot\mathbf{E}\times\mathbf{H}=-\mathbf{E}\cdot\mathbf{J}_0-\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}-\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}$$

$$-\mathbf{E}\cdot\mathbf{J}_0=\frac{\partial}{\partial t}\left(\frac{1}{2}\,\varepsilon_0E^2\right)+\frac{\partial}{\partial t}\left(\frac{1}{2}\,\mu_0H^2\right)+\nabla\cdot\left(\mathbf{E}\times\mathbf{H}\right)$$

Performing volume integration on both sides, and using the divergence theorem for the last term on the right side, we get

$$-\int_V \left(\mathbf{E} \cdot \mathbf{J}_0\right) dv = \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\,\varepsilon_0 E^2\right) dv + \frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\,\mu_0 H^2\right) dv + \oint_S \mathbf{P} \cdot d\mathbf{S}$$

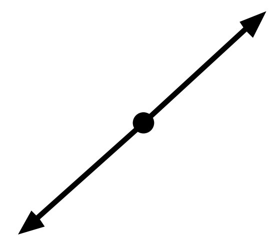
where we have defined $\mathbf{P} = \mathbf{E} \times \mathbf{H}$, known as the Poynting vector. The equation is known as the Poynting's Theorem.

Polarization

The characteristic which describes how the position of the tip of the vector varies with time.

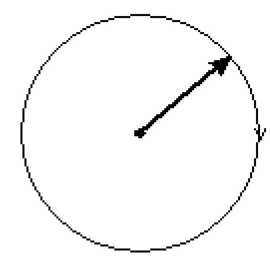
Linear Polarization:

Tip of the vector describes a line.



Circular Polarization:

Tip of the vector describes a circle.



SKIN EFFECT

- The effect of resistance in a transmission line is to continuously reduce the amplitude of both incident and reflected voltage and current waves.
- **Skin Effect:** As frequency increases, depth of penetration into adjacent conductive surfaces decreases for boundary currents associated with electromagnetic waves. This results in the confinement of the voltage and current waves at the boundary of the transmission line, thus making the transmission more lossy.
- The skin depth is given by:

$$\text{skin depth (m)} = 1/(\pi\mu\gamma f)^{0.5}$$

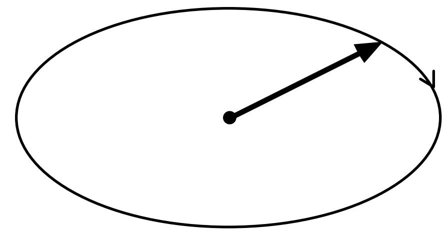
where f = frequency, Hz

μ = permeability, H/m

γ = conductivity, S/m

Elliptical Polarization:

- Tip of the vector describes an ellipse.



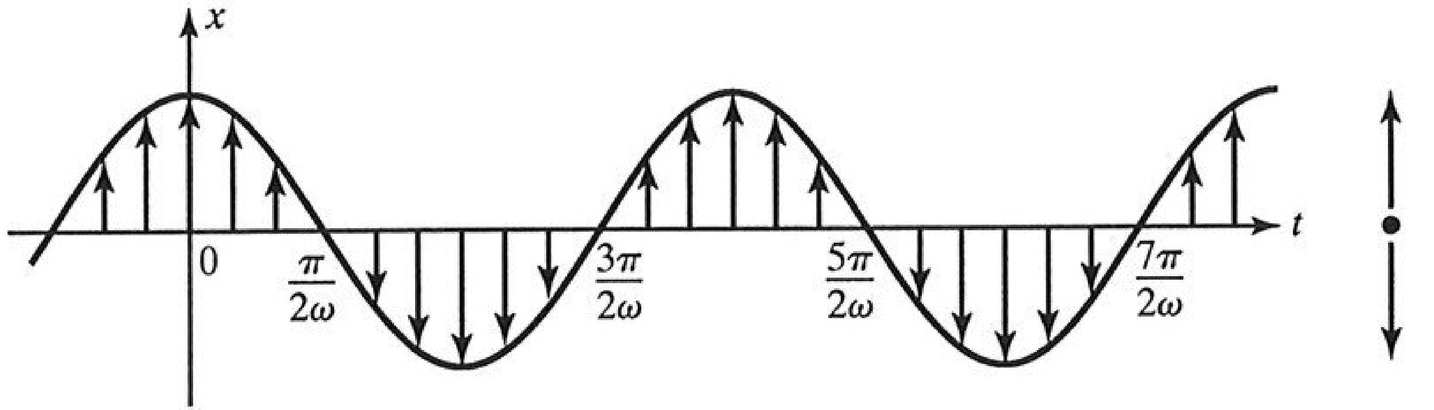
(i) **Linear Polarization**

$$\mathbf{F}_1 = \underbrace{F_1 \cos(\omega t + \phi)}_{\text{Magnitude varies sinusoidally with time}}\, \mathbf{a}_x$$

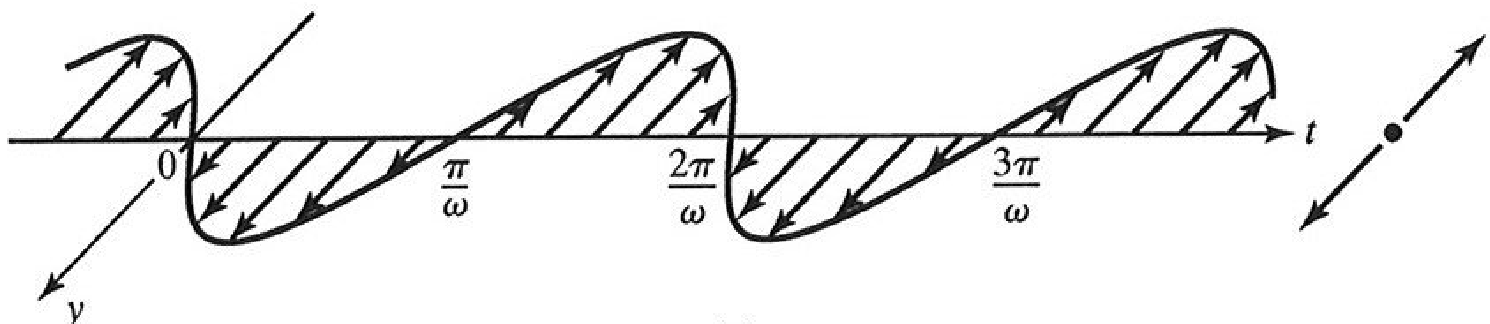
Direction remains along the *x* axis

Linearly polarized in the *x* direction.

Linear polarization



(a)



(b)

$$\mathbf{F}_2 = \underbrace{F_2 \cos(\omega t + \theta)}_{\text{Magnitude varies sinusoidally with time}} \mathbf{a}_y$$

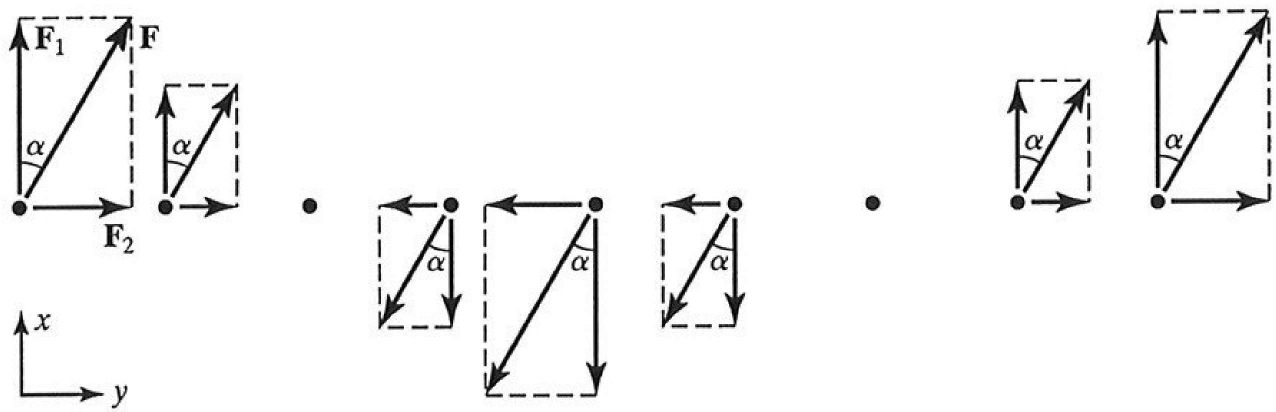
Direction remains along the *y* axis

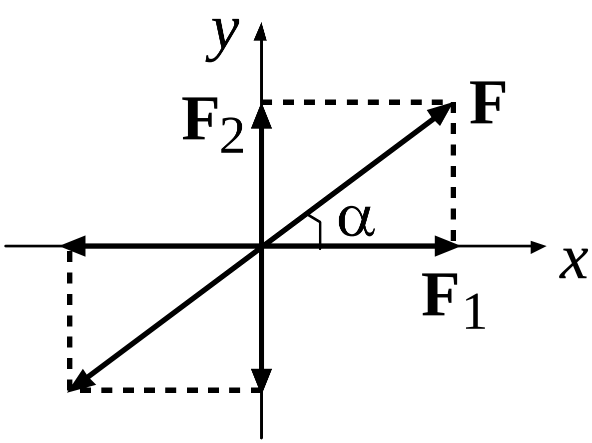
Linearly polarized in the *y* direction.

If two (or more) component linearly polarized vectors are in phase, (or in phase opposition), then their sum vector is also linearly polarized.

Ex: $\mathbf{F} = F_1 \cos(\omega t + \phi)\, \mathbf{a}_x + F_2 \cos(\omega t + \phi)\, \mathbf{a}_y$

Sum of two linearly polarized vectors in phase is a linearly polarized vector





$$\alpha = \tan^{-1} \frac{F_2 \cos(\omega t + \phi)}{F_1 \cos(\omega t + \phi)}$$

$$= \tan^{-1} \frac{F_2}{F_1}$$

$$= \text{constant}$$

Circular Polarization

If two component linearly polarized vectors are

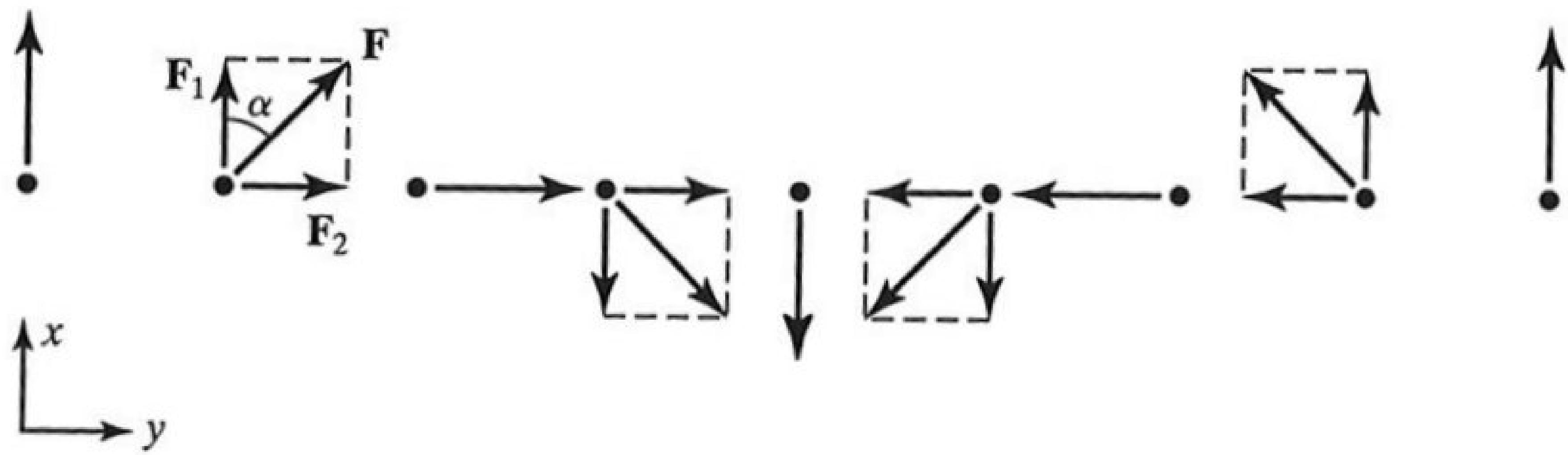
(a) equal to amplitude

(b) differ in direction by 90˚

(c) differ in phase by 90˚,

then their sum vector is circularly polarized.

Circular Polarization



Example:

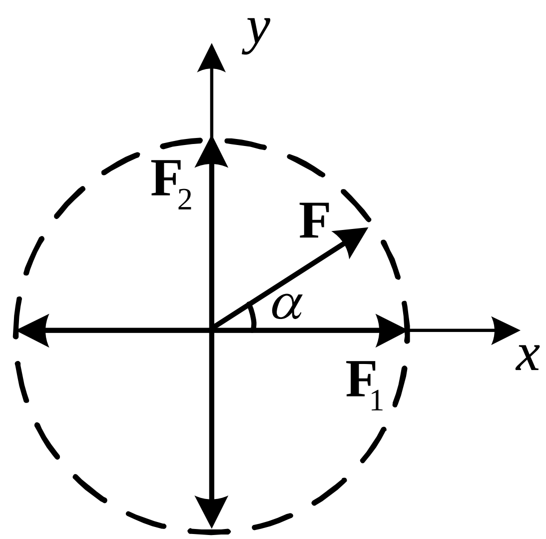
$$\mathbf{F} = F_1 \cos \omega t \, \mathbf{a}_x + F_1 \sin \omega t \, \mathbf{a}_y$$

$$|\mathbf{F}| = \sqrt{(F_1 \cos \omega t)^2 + (F_1 \sin \omega t)^2}$$

$$= F_1, \text{ constant}$$

$$\alpha = \tan^{-1} \frac{F_1 \sin \omega t}{F_1 \cos \omega t}$$

$$= \tan^{-1}(\tan \omega t) = \omega t$$

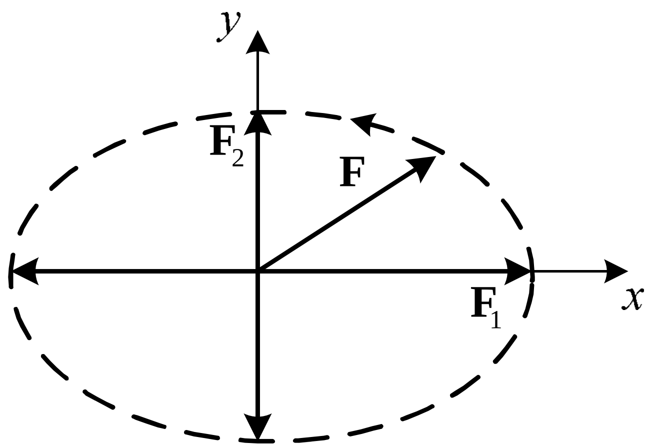


Elliptical Polarization

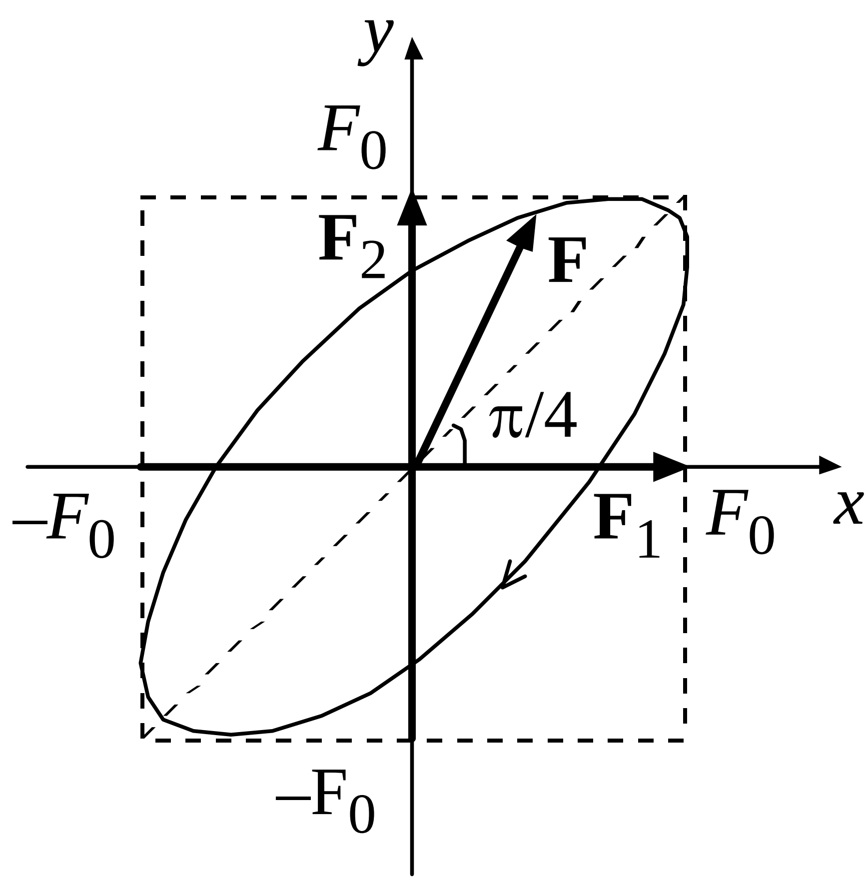
In the general case in which either of (i) or (ii) is not satisfied, then the sum of the two component linearly polarized vectors is an elliptically polarized vector.

Example:

$$\mathbf{F} = F_1 \cos \omega t \, \mathbf{a}_x + F_2 \sin \omega t \, \mathbf{a}_y$$



Example: $\mathbf{F} = F_0 \cos \omega t \; \mathbf{a}_x + F_0 \cos\left(\omega t + \pi/4\right) \; \mathbf{a}_y$



UNIT-4

Transmission and reflection of planes waves at the boundaries

Plane Wave Propagation at Arbitrary Angle

Plane waves are not normally incident, so now we must consider the general problem of a plane wave propagating along a specified axis that is arbitrarily relative to a rectangular coordinate system. The most convenient way is in terms of the direction cosines of the uniform plane wave, the equiphase surfaces are planes perpendicular to the direction of propagation.

Definitions:

uniform planes – a free space plane wave at an infinite distance from the generator, having constant amplitude electric and magnetic field vectors over the equiphase surfaces.

equiphase surface – any surface in a wave over which the field vectors of a particular instant have either 0° or 180° phase difference.

For a plane wave propagating along the +z axis

$$\hat{\mathrm{E}}(z) = \hat{\mathrm{E}}_m e^{-j\beta z} a_x \qquad (6.1)$$

Equation (6.1) states that each z equal to a constant plane will represent an equiphase surface with no spatial variation in the electric or magnetic fields. In other words,

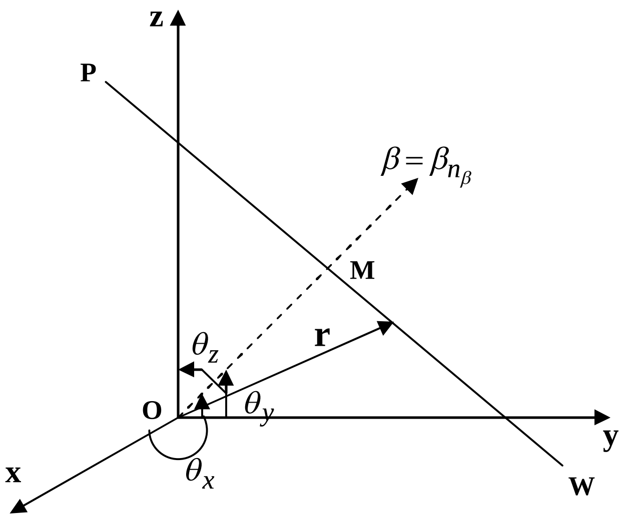
$$\frac{\partial}{\partial_x} = 0 = \frac{\partial}{\partial_y} \quad \Rightarrow \text{for a uniform plane wave}$$

It will be necessary to replace **z** for a plane wave traveling in an arbitrary direction with an expression when put equal to a constant (βz = constant), that will result in equiphase surfaces.

The equation of an equiphase plane is given by

$$\beta \cdot r = \beta\, n_\beta \cdot r$$

The radial vector (r) from the origin to any point on the plane, and β is the vector normal to the plane is shown in Figure (6.1).



As you can see from figure 6.1, the plane perpendicular to the vector **β** is seen from its side appearing as a line P-W. The dot product $\mathbf{n}_\beta \cdot$ r is the projection of the radial vector **r** along the normal to the plane and will have the constant value **OM** for all points on the plane. The equation **β · r = constant** is the characteristic property of a plane perpendicular to the direction of propagation **β**.

The equiphase equation is

$$\beta \cdot r = \beta_x x + \beta_y y + \beta_z z$$

$$= \beta (\cos \theta_x x + \cos \theta_y y + \cos \theta_z z)$$

$$= \text{constant}$$

$$\mathbf{r} = x a_x + y a_y + z a_z$$

$$\beta = \beta_x \boldsymbol{a}_x + \beta_y \boldsymbol{a}_y + \beta_z \boldsymbol{a}_z$$

θ_x, θ_y, θ_z, are the angles the **β** vector makes with x, y, and z axes, respectively.

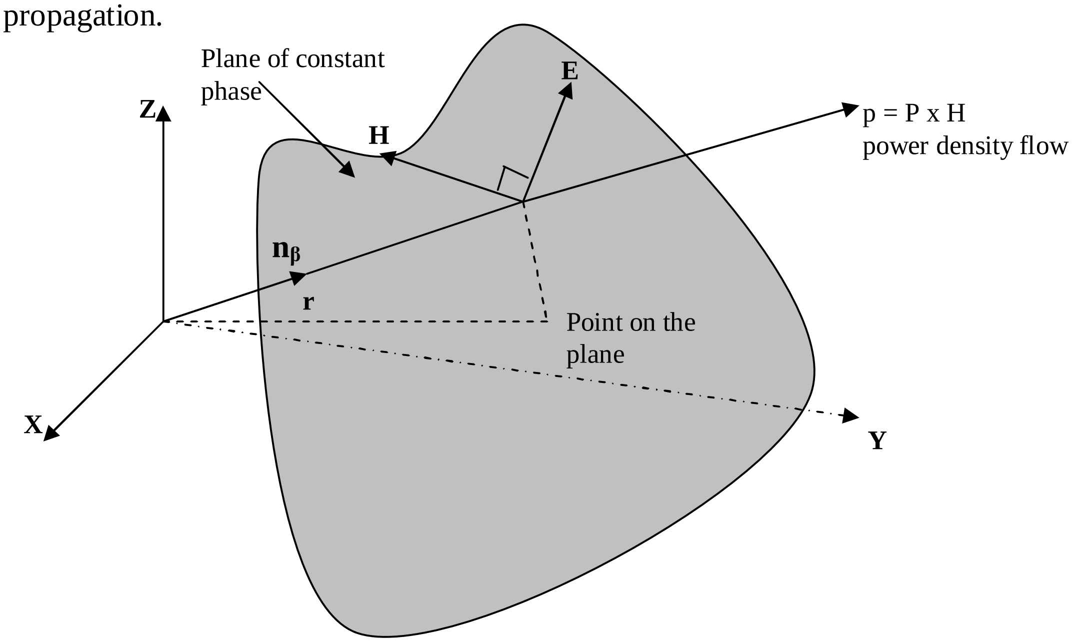
Definition:

transverse electromagnetic wave (TEM) – electromagnetic wave having electric field vectors and magnetic field vectors perpendicular to the direction of propagation.

H is perpendicular to **E**, and both **E** and **H** are perpendicular to the direction of propagation **β**. The expressions for $\hat{E}$ *and* $\hat{H}$ are

$$\hat{E} = \hat{E}_m e^{-j\beta \cdot r}$$
$$\hat{H} = \frac{n_\beta \wedge \hat{E}}{\eta} \tag{6.2}$$

The unit vector $\mathbf{n}_\beta$ along **β** and **η** is the wave impedance in the propagation medium. See Figure 6.2 for the illustration of orthogonal relations between $\hat{E}$ *and* $\hat{H}$ and the direction of propagation.



EXAMPLE 6.1

The vector amplitude of an electric field associated with a plane wave that propagates in the negative **z** direction in free space is given by $\hat{E}_m = 2a_x + 3a_y \, V/m$

Find the magnetic field strength.

Solution:

The direction of propagation $\mathbf{n}_\beta$ is $-\mathbf{a}_z$. The vector amplitude of the magnetic field is then given by

$$\hat{H}_m = \frac{n_\beta \wedge \hat{E}}{\eta} = \frac{1}{\eta_\circ}\begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & -1 \\ 2 & 3 & 0 \end{vmatrix} = \left(\frac{1}{377} 3a_x - 2a_y\right) A/m$$

***note** $\eta_\circ = \sqrt{\dfrac{\mu_\circ}{\varepsilon_\circ}}$ $120\pi \sim 377\Omega$ (Appendix D – Table D.1)

Reflection by Perfect Conductor – Arbitrary Angle of Incidence

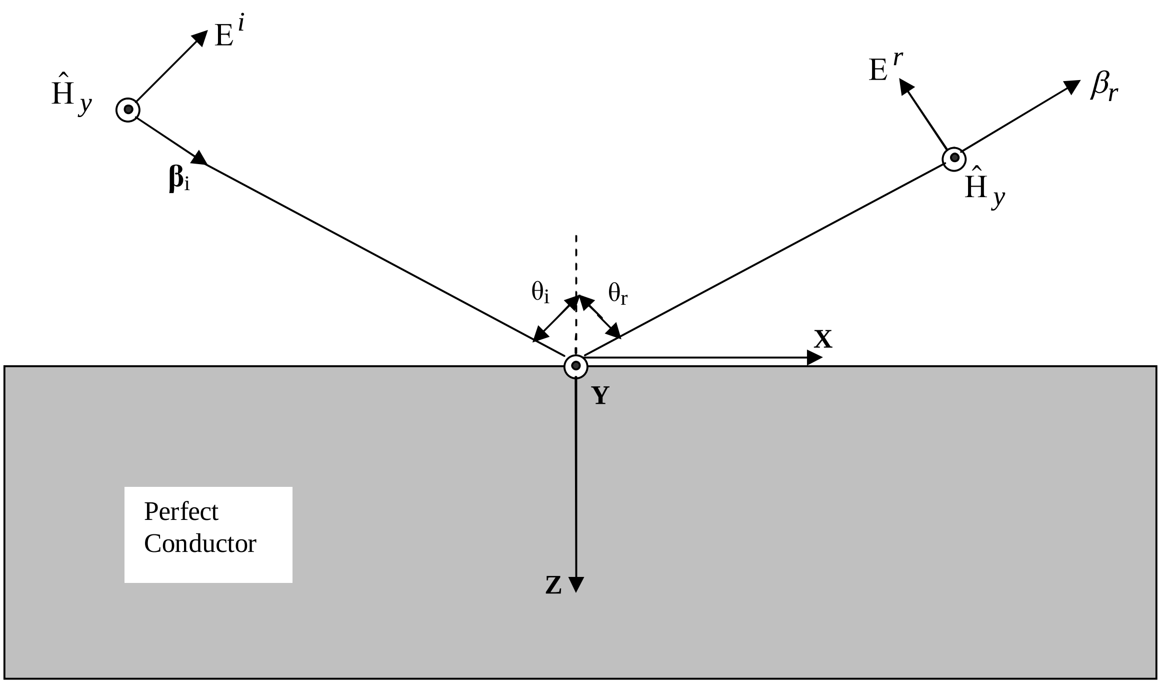
By decomposing the general problem into two special cases we can simplify our analysis.

1. E field is polarized in the plane formed by the normal to the reflecting surface in the direction β_i of the incident wave.
2. E field is perpendicular to the plane of incidence.

The plane formed by the normal to the reflecting surface and the direction of propagation $\boldsymbol{\beta}$ is known as the plane of incidence. The general case can be considered as a superposition of two cases –

- E is parallel to the plane of incidence
- E is perpendicular to the plane of incidence

E Field Parallel to Plane of Incidence



The figure shows an incident wave polarized with the E field in the plane of incidence and the power flow in the direction of β_i at angle θ_i with respect to the normal to the surface of the perfect conductor.

The direction of propagation is given by the Poynting vector and the β_i , **E,** and **H** fields need to be arranged so that β_i is in the same direction as $\mathrm{E}^i \wedge \mathrm{H}^i$ at any time. The magnetic field is out of the plane of the paper, $\mathrm{H} = \hat{\mathrm{H}}_y\, a_y$ for the direction of the electric field shown. There is no transmitted field within the perfect conductor; however there will be a reflected field with power flow at the angle θ_r with respect to the normal to the interface. To maintain the power density flow $\mathrm{E}^r \wedge \mathrm{H}^r$ will be in the same direction β_r as. The expression for the total electric field in free space is

$$\hat{\mathrm{E}} = \hat{\mathrm{E}}^i + \hat{\mathrm{E}}^r = \hat{\mathrm{E}}_m^i\, e^{-j\beta_i \cdot r} + \hat{\mathrm{E}}_m^r\, e^{-j\beta_r \cdot r} \qquad (6.5)$$

$$\beta_i \cdot r = \beta\left(\cos\theta_i\, a_z + \sin\theta_i\, a_x\right)\cdot\left(x\, a_x + y\, a_y + z\, a_z\right)$$

$$= \beta\left(x\sin\theta_i + z\cos\theta_i\right) \qquad (6.6)$$

$$\beta_r \cdot r = \beta\left(x\sin\theta_r - z\cos\theta_r\right) \qquad (6.7)$$

The total electric field has **x** and **z** components:

$$\hat{\mathrm{E}}_x(x,z) = \hat{\mathrm{E}}_m^i \cos\theta_i\, e^{-j\beta_i \cdot r} + \hat{\mathrm{E}}_m^r \cos\theta_r\, e^{-j\beta_r \cdot r}$$

$$\hat{\mathrm{E}}_z(x,z) = -\hat{\mathrm{E}}_m^i \sin\theta_i\, e^{-j\beta_i \cdot r} + \hat{\mathrm{E}}_m^r \sin\theta_r\, e^{-j\beta_r \cdot r}$$

$$\left.\hat{\mathrm{E}}_x\right|_{at\, z=0} = \hat{\mathrm{E}}_m^i \cos\theta_i\, e^{-j\beta_i \cdot r} - \hat{\mathrm{E}}_m^r \cos\theta_r\, e^{-j\beta_r \cdot r} = 0$$

$$= \hat{\mathrm{E}}_m^i \cos\theta_i\, e^{-j\beta x \cdot \sin\theta_i} - \hat{\mathrm{E}}_m^r \cos\theta_r\, e^{-j\beta x \cdot \sin\theta_r} = 0 \qquad (6.8)$$

Equation 6.8 shows the relationship between the incident and reflected amplitudes for a perfect conductor the total tangential **E** field at the surface must be zero which satisfies the boundary condition. To be zero at all values of x along the surface of the conducting plane, the phase terms must be equal to each other –

$$\theta_{i=}\,\theta_r \qquad (6.9)$$

Equation 6.9 is known as Snell's law of reflection.

Definition:

Snell's Law is a rule of Physics that applies to visible light passing from air (or vacuum) to some medium with an index of refraction different from air.

Substitute equation 6.9 into equation 6.8 –

$$\hat{E}_m^i = \hat{E}_m^r \tag{6.10}$$

Therefore, the total electric field in free space is

$$\hat{E}(x,z) = \hat{E}_x(x,z)a_x + \hat{E}_z(x,z)a_z$$

$$= \hat{E}_{mi}\cos\theta_i e^{-j\beta x\sin\theta_i}\left(e^{-j\beta z\cos\theta_i} - e^{j\beta z\cos\theta_i}\right)a_x$$
$$- \hat{E}_m^i \sin\theta_i e^{-j\beta x\sin\theta_i}\left(e^{-j\beta z\cos\theta_i} + e^{j\beta z\cos\theta_i}\right)a_z$$

$$= 2j\hat{E}_m^i \cos\theta_i \sin\left(\beta z\cos\theta_i\right)e^{-j\beta x\sin\theta_i} a_x \tag{6.11}$$

$$- 2j\hat{E}_m^i \sin\theta_i \cos\left(\beta z \cos\theta_i\right)e^{-j\beta x\sin\theta_i} a_z$$

$$= 2\hat{E}_m^i[-j\cos\theta_i \sin\left(\beta z\cos\theta_i\right) a_x$$

$$- \sin\theta_i \cos\left(\beta z\cos\theta_i\right)a_z]e^{-j\beta x\sin\theta_i}$$

Take equation 6.11 and recover the time-domain form of the total electric field

$$E(r,t) = \mathrm{Re}\left(\hat{E}(r)e^{j\varpi t}\right)$$

Observe the variation of the total field with the **x** variable indicating there is a traveling wave in the **x** direction with a phase constant

$$\beta_x = \beta\sin\theta_i$$

And in the **z** direction the field forms a standing wave.

The total magnetic field is

$$\hat{\mathrm{H}}(x,z) = \hat{\mathrm{H}}_y\,(x,z)\,a_y = \hat{\mathrm{H}}_y^i\,(x,z)\,a_y + \hat{\mathrm{H}}_y^r\,(x,z)\,a_y$$

Use the relation $\mathrm{H} = \dfrac{n_\beta \wedge \mathrm{E}}{\eta}$ for each of the incident and reflected fields to employ the expressions **x** and **z** components of the incident and reflected electric fields.

$$\hat{\mathrm{H}}^i = \frac{n_{\beta i}}{\eta} \wedge \hat{\mathrm{E}}^i$$

$$= \frac{1}{\eta}\begin{vmatrix} a_x & a_y & a_z \\ \sin\theta_i & 0 & \cos\theta_i \\ \hat{\mathrm{E}}_m^i \cos\theta_i\, e^{-j\beta(\sin\theta_i x + \cos\theta_i z)} & 0 & -\hat{\mathrm{E}}_m^i \sin\theta_{i_i}\, e^{-j\beta(\sin\theta_i x + \cos\theta_i z} \end{vmatrix}$$

The solution of the determinant, the only nonzero component of $\hat{\mathrm{H}}^i$ is the $\mathbf{a}_\mathbf{y}$ component given by

$$\hat{\mathrm{H}}^i = \frac{1}{\eta} a_y \left[\hat{\mathrm{E}}_m^i \cos^2\theta_i\, e^{-j\beta\left(\sin\theta_i x + \cos\theta_i z\right)} + \hat{\mathrm{E}}_m^i \sin^2\theta_i\, e^{-j\beta\left(\sin\theta_i x + \cos\theta_i z\right)}\right]$$

$$= \frac{\hat{\mathrm{E}}_m^i}{\eta} e^{-j\beta\left(\sin\theta_i x + \cos\theta_i z\right)} a_y$$

The reflected magnetic fields is given by

$$\hat{\mathrm{H}}^r = \frac{\hat{\mathrm{E}}_m^i}{\eta} e^{-j\beta\left(\sin\theta_i x - \cos\theta_i z\right)} a_y$$

The total magnetic field $\hat{\mathrm{H}}$ (x, z) is

$$\hat{\mathrm{H}}\,(x,z) = a_y \frac{2\hat{\mathrm{E}}_m^i}{\eta} \cos\left(\beta\, z \cos\,\theta_i\right) e^{-j\beta x \sin\theta_i}$$

The average power flow parallel to the conducting surface is

$$\mathrm{P}_{ave}(x,z) = \frac{1}{2}\,\mathrm{Re}\left[\hat{\mathrm{E}} \wedge \hat{\mathrm{H}}\right]$$

$$= \frac{1}{2}\,\mathrm{Re}\begin{vmatrix} a_x & a_y & a_z \\ \hat{\mathrm{E}}_x & 0 & \hat{\mathrm{E}}_z \\ 0 & \hat{\mathrm{H}}_y^* & 0 \end{vmatrix}$$

The cross product yields two components:

- One in the **x** direction
- One in the **z** direction

$$\mathrm{P}_{ave} = \frac{1}{2}\,\mathrm{Re}\left[-\hat{\mathrm{E}}_z\hat{\mathrm{H}}_y^*\, a_x + \hat{\mathrm{E}}_x\hat{\mathrm{H}}_y^*\, a_z\right]$$

The expression of $\mathrm{P}_{\mathrm{ave}}$ will reduce to

$$\mathrm{P}_{ave}(x,z) = \frac{1}{2}\,\mathrm{Re}\left[-\hat{\mathrm{E}}_z\hat{\mathrm{H}}_y^*\right] a_x$$

$$\frac{2\left|\hat{\mathrm{E}}_m^i\right|}{\eta}\sin\theta_i\cos^2\left[\beta\; z\cos\theta_i\right] a_x$$

Glancing Incident:

$\left(\theta_i \to 90^\circ\right)$, $\mathrm{P}_{ave} = \dfrac{\left(2\left(\hat{\mathrm{E}}_m^i\right)^2\right)}{\eta} a_x$, the power flow is at maximum.

Normal Incident:

$\theta_i = 0$, $\mathrm{P}_{x,ave} = 0$ (Power flow in the **x** direction is zero)

Average power flow perpendicular to the conducting surface is zero, because the average Poynting Vector is zero in that direction

$$P_{z,ave} = \frac{1}{2}\,\mathrm{Re}\left(\hat{\mathrm{E}}_x\,\hat{\mathrm{H}}_y^*\right) = 0$$

Why? Because $\hat{E}_x$ is multiplied by **j**, therefore $\hat{E}_x$ *and* $\hat{H}_y$ are out of phase by **90°**. Therefore, a traveling-wave pattern occurs in the **x** direction, because the incident and reflected waves travel in the same direction, the standing-wave pattern will be observed in the **z** direction, because the incident and reflected waves travel in the opposite directions.

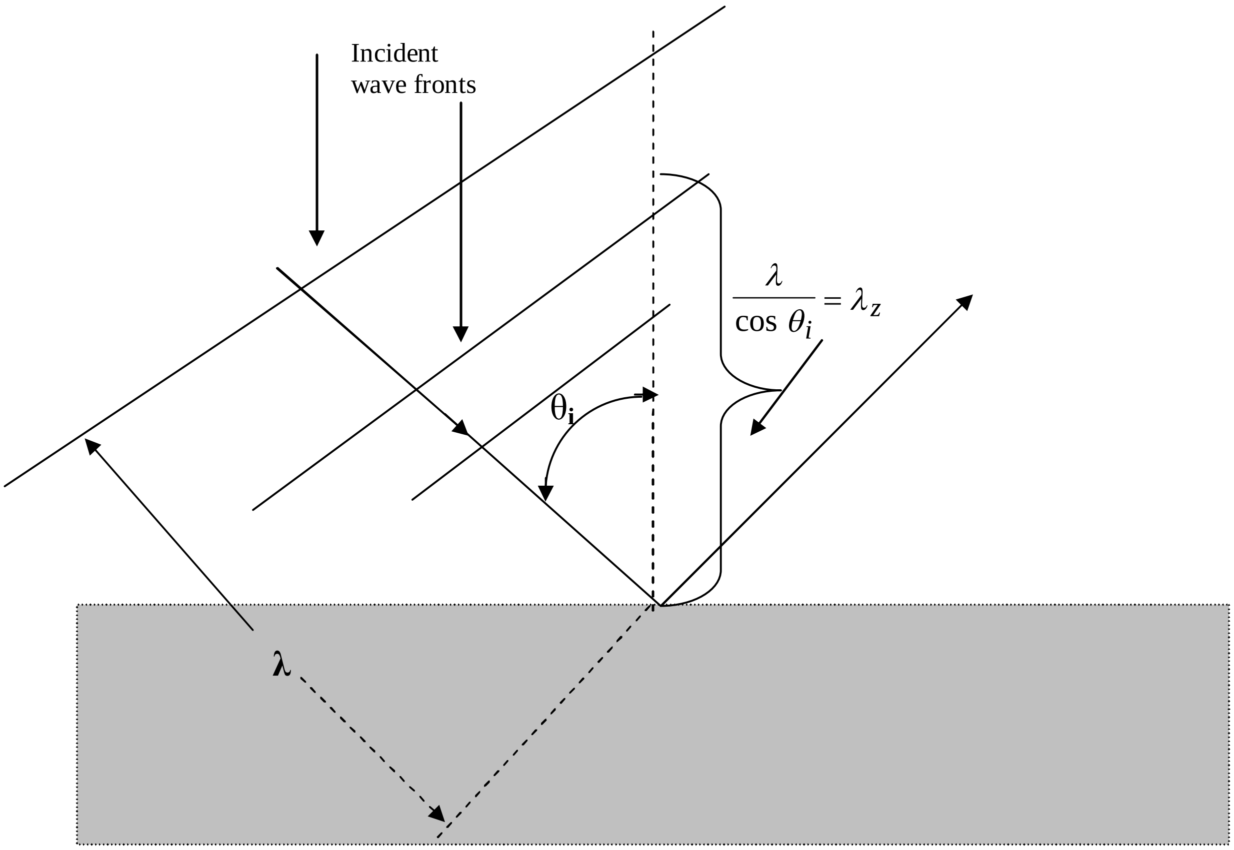
The location of zeros (nodes) of the $\hat{E}_x$ field can be found by letting $\sin\left(\beta z \cos\theta_i\right) = 0$. At a distance **z** from the conducting plane given by

$$\beta z \cos\theta_i = n\pi$$

Or

$$z = n\,\frac{\lambda}{2\cos\,\theta_i} \qquad n = 0, 1, 2, \ldots$$

The zeros will occur at distances larger than integer multiples of $\lambda/2$. So, for normal incidence, $\theta_i = 0$, $\cos\theta_i = 1$, and the positions of the zeros will are the same as those discussed in chapter 5. For the oblique incidence, the locations of the standing-wave nodes are $\lambda/2$ apart along the direction of propagation. The wavelength measured along the **z**-axis is greater than the wavelength of the incident waves along the direction of propagation. As shown in Figure 6.4 the relation between these wavelengths is $\lambda_z = \dfrac{\lambda}{\cos\theta_i}$.



The plane of the zero $\hat{E}_x$ field occur at multiples of $\lambda/2$ along the direction of propagation, and they are located at integer multiples of $\lambda_z/2$ along the z-axis which appear separated by larger distances. Also note that the standing-wave pattern associated with the $\hat{E}_z$ component may appear as if there is no zero value of the electric field at z = 0, but the $\hat{E}_z$ component is normal to the reflecting surface, therefore the boundary condition is not in violation.

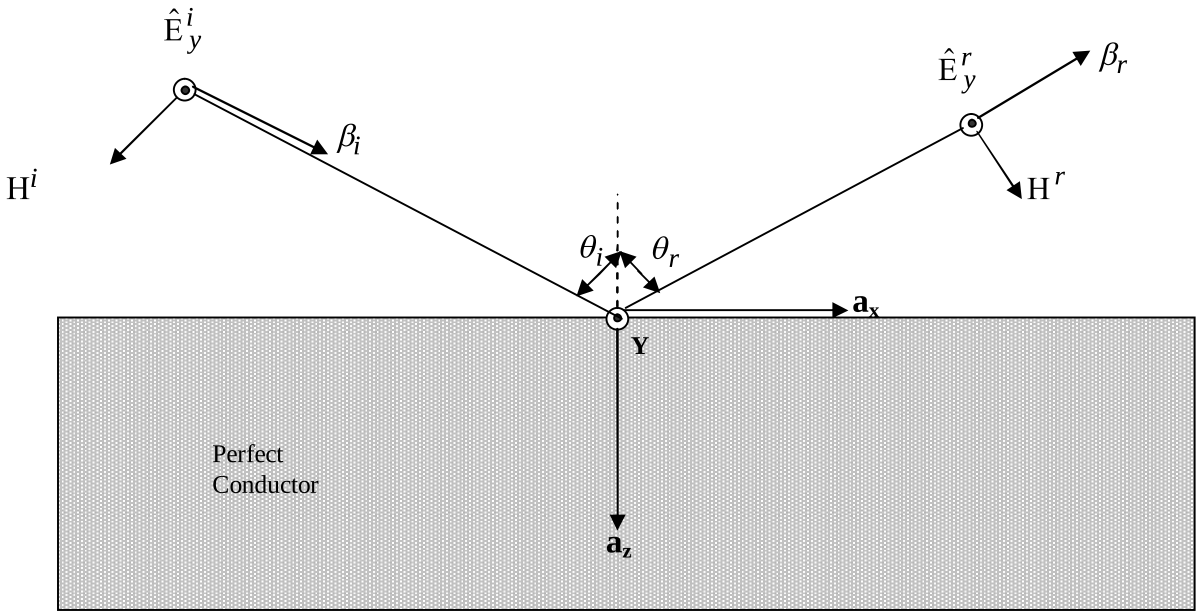
Electric Field Normal to the Plane of Incidence

The entire electric field is (out of the paper) in the **y** direction and the magnetic field will have both **x** and **z** components. See Figure 6.5.

The incident electric and magnetic fields are

$$\hat{E}^i = \hat{E}^i_m e^{-j\beta_i \cdot r}$$

$$\hat{H}^i = \frac{n_{\beta_i} \wedge \hat{E}^i}{\eta} = \frac{\hat{E}^i_m}{\eta}\left(-\cos\theta_i a_x + \sin\theta_i a_z\right) e^{-j\beta_i \cdot r}$$



where $\beta_i \cdot r = \beta\left(\sin\theta_i\, x - \cos\theta_i\, z\right)$. Assume that the reflected field is also in the **y** direction so the magnetic field must be perpendicular to both **E** and the Poynting Vector **P = E ^ H**,

$$\hat{E}^r = \hat{E}_m^r e^{-j\beta_r \cdot r} a_y$$

$$\hat{H}^r = \frac{n_{\beta_r} \wedge \hat{E}^r}{\eta} = \frac{\hat{\hat{E}}_m^r}{\eta}\left(\cos\theta_r\, a_x + \sin\theta_r\, a_z\right) e^{-j\beta_r \cdot r}$$

Where $\beta_r \cdot r = \beta\left(\sin\theta_r\, x - \cos\theta_r\, z\right)$. Determine the angle of reflection θ_r and the amplitude of the reflected electric field $\hat{E}_m^r$ by using the boundary conditions at z = 0. This also includes zero values of the tangential electrical field **E** and the normal component of the magnetic field **H**.

$$\hat{E}_y(x,z) = \hat{E}_y^i + \hat{E}_y^r = 0 \qquad \text{at } z = 0$$

Therefore,

$$\hat{E}_y(x,0) = \hat{E}_m^i e^{-j\beta x \sin\theta_i} + \hat{E}_m^r e^{-j\beta x \sin\theta_r} = 0$$

And

$$\hat{H}_z(x,0) = \frac{1}{\eta}\hat{E}_m^i \sin\theta_i e^{-j\beta x \sin\theta_i} + \frac{1}{\eta}\hat{E}_m^r \sin\theta_i e^{-j\beta x \sin\theta_r} = 0$$

Note: These two conditions will provide the same results for the unknowns $\theta_r \; and \hat{E}_m^r$, and be true for every value of **x** along **z = 0** plane, so the phase factors must be equal.

$$\theta_r = \theta_i$$

And

$$\hat{E}_m^r = -\hat{E}_m^i$$

Negative sign indicates the opposite direction of the reflected electric field (i.e. into the paper)

The total **E** field is

$$\hat{E}_y(x, z) = \hat{E}_m^i e^{-j\beta x \sin\theta_i}\left(e^{-j\beta z \cos\theta_i} - e^{j\beta z \cos\theta_i}\right)$$
$$= -2j\hat{E}_m^i\left[\sin\left(\beta z \cos\theta_i\right)\right]e^{-j\beta x \sin\theta_i}$$

The total **H** field is

$$\hat{H} = \hat{H}^i + \hat{H}^r = \left[\frac{n_{\beta_i}}{\eta} \wedge a_y \hat{E}_m^i e^{-j\beta_i \cdot r}\right] - \left[\frac{n_{\beta_i}}{\eta} \wedge a_y \hat{E}_m^i e^{-j\beta_i \cdot r}\right]$$

And the substitution of $\hat{E}_m^r = \hat{E}_m^i$ has been made. The direction vectors of the incident and reflective wave are

$$n_{\beta i,\, r} = \sin\theta_i a_x \pm \cos\theta_i a_z$$

And

$$n_{\beta i, r} \wedge a_y = \sin\theta_i a_z \mp \cos\theta_i a_x$$

The components of the total magnetic field are

$$\hat{H}_x(x,z) = \frac{-2\hat{E}_m^i}{\eta}\cos\theta_i \cos\left(\beta z \cos\theta_i\right)e^{-j\beta x \sin\theta_i}$$

$$\ddot{H}_z(x, z) = \frac{-2j\hat{E}_m^i}{\eta} \sin\theta_i \sin(\beta z \cos\theta_i)\ e^{-j\beta x \sin\theta_i}$$

There is a standing-wave in the **z** direction because the reflected and incident waves travel in the opposite direction along the z-axis. The fields traveling in the **x** direction and having the only nonzero power flow in the direction parallel to the interface.

The concept can be illustrated by considering the average density flow associated with the wave.

$$P_{ave}(x, z) = \frac{1}{2} \operatorname{Re}\left[\hat{E} \wedge \hat{H}^*\right]$$

$$= \frac{1}{2} \operatorname{Re} \begin{vmatrix} a_x & a_y & az \\ 0 & T & 0 \\ S & 0 & W \end{vmatrix}$$

$$= \frac{2\left|\hat{E}_m^i\right|}{\eta} \sin\theta_i \sin^2(\beta z \cos\theta_i)\ a_x$$ $\Rightarrow$ This indicates that the power flow is in the x direction.

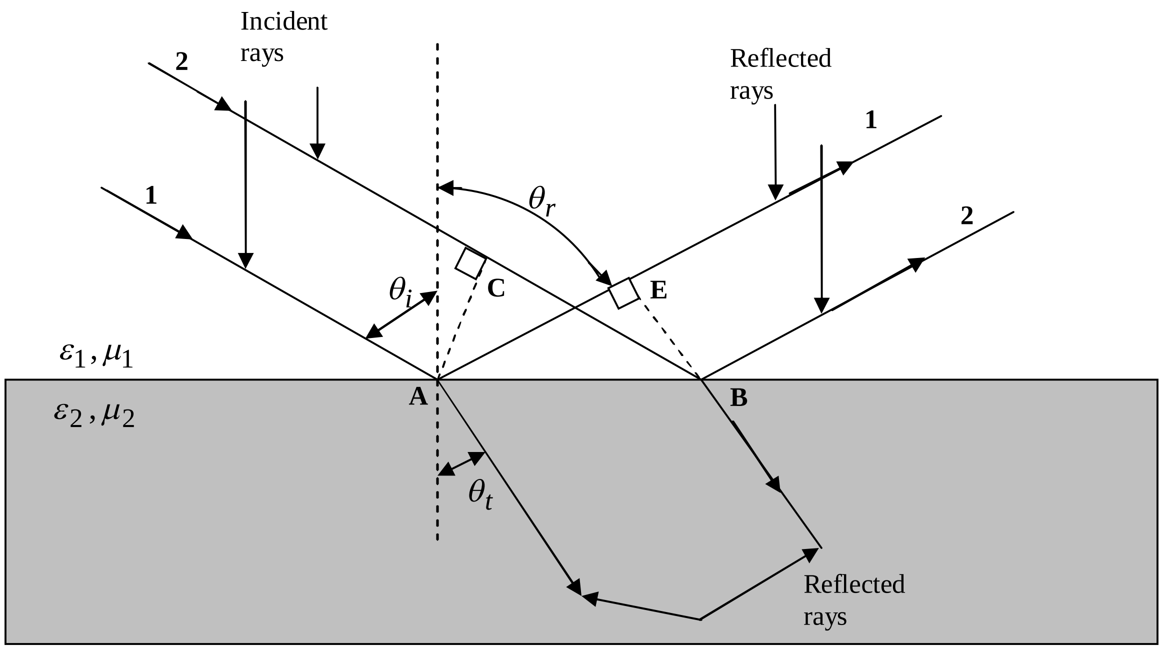
$$S = \frac{-2\hat{E}_m^i}{\eta} \cos\theta_i \cos(\beta z \cos\theta_i)\, e^{+j\beta x \sin\theta_i},\quad T = -2\hat{E}_m^i \sin(\beta z \cos\theta_i)\, e^{-j\beta x \sin\theta_i},$$

$$W = \frac{+2\hat{E}_m^{i*}}{\eta} \sin\theta_i \sin(\beta z \cos\theta_i)\, e^{+j\beta x \sin\theta_i}$$

Reflection and Refraction at Plane Interface between Two Media:

Oblique Incidence

Figure 6.7 shows two media with electrical properties ε_1 and μ_1 in medium 1, and ε_2 and μ_2 in medium 2. Here a plane wave incident angle θ_i on a boundary between the two media will be partially transmitted into and partially reflected at the dielectric surface. The transmitted wave is reflected into the second medium, so its direction of propagation is different from the incidence wave. The figure also shows two rays for each the incident, reflected, and transmitted waves. A ray is a line drawn normal to the equiphase surfaces, and the line is along the direction of propagation.



The incident ray 2 travels the distance CB, while on the contrary the reflected ray 1 travels the distance AE. For both AC and BE to be the incident and reflected wave fronts or planes of equiphase, the incident wave should take the same time to cover the distance AE. The reason being that the incident and reflected wave rays are located in the same medium, therefore their velocities will be equal,

$$\frac{CB}{V_1} = \frac{AE}{V_2}$$

OR

$$AB \sin\theta_i = AB \sin\theta_r$$

With this being the case then it follows that

$$\theta_i = \theta_r$$

What is the relationship between the angles of incidence θ_i and refraction θr ?

It takes the incident ray the equal amount of time to cover distance CB as it takes the refracted ray to cover distance AD –

$$\frac{CB}{V_1} = \frac{AD}{V_2}$$

And the magnitude of the velocity V_1 in medium 1 is:

$$V_1 = \frac{1}{\sqrt{\mu_1 * \varepsilon_1}}$$

And in medium 2:

$$V_2 = \frac{1}{\sqrt{\mu_2 * \varepsilon_2}}$$

Also,

$$CB = AB \sin\theta_i$$
$$AD = AB \sin\theta_i$$

Therefore,

$$\frac{CB}{AD} = \frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2} = \sqrt{\frac{\mu_2 * \varepsilon_2}{\mu_1 * \varepsilon_1}}$$

For most dielectrics $\mu_2 = \mu_1 = \mu_\circ$

Therefore, $$\frac{\sin\theta_i}{\sin\theta_t} = \left.\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right|_{\mu_1 = \mu_2 = \mu_\circ} \qquad (6.12)$$

Equation 6.12 is known as Snell's Law of Refraction.

Parallel Polarization Case – E is in Plane of Incidence

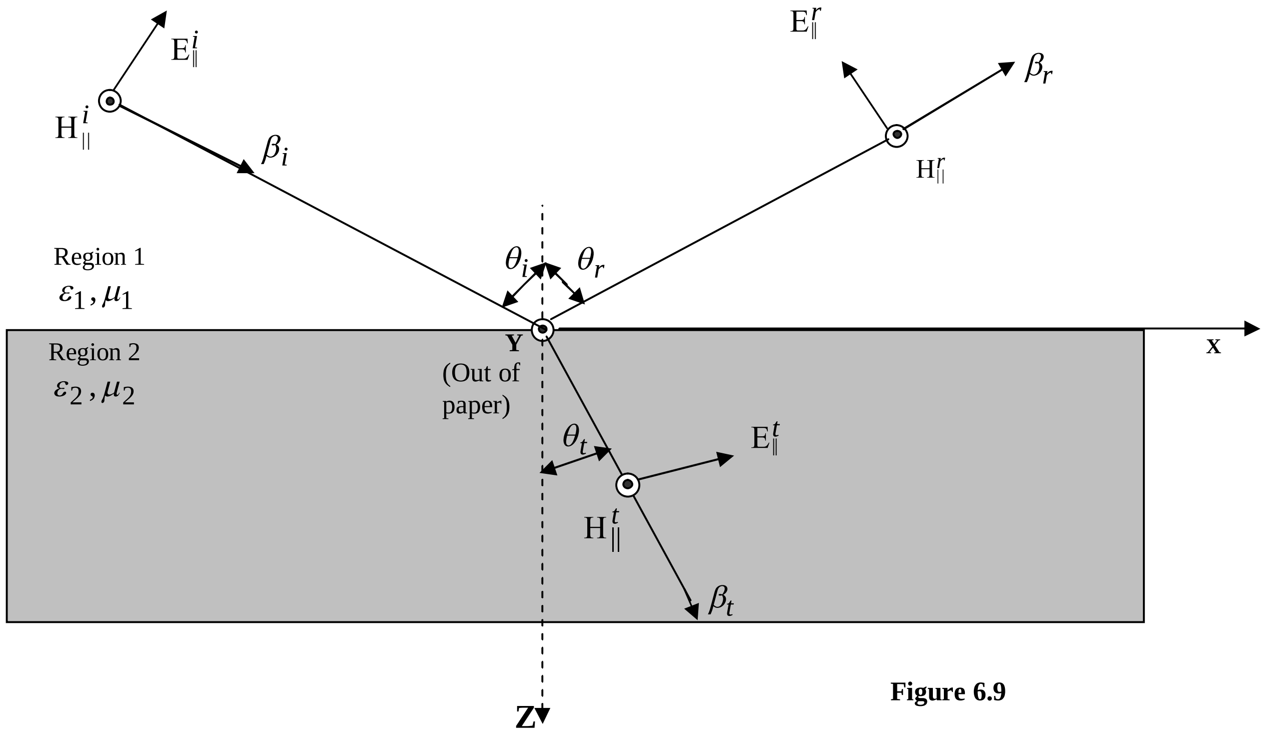


Figure 6.9

The unknown amplitudes of the reflected and transmitted electric fields $E_{\|}^{r}$ *and* $E_{\|}^{t}$ can be determined by simply applying the boundary conditions at the dielectric interface. The electric fields $E_{\|}^{r}$ *and* $E_{\|}^{t}$ will now be used in the analysis to emphasize the case of parallel polarization, instead of using the electric fields E_{m}^{r} *and* E_{m}^{t}.

The tangential component of **H** should be continuous across the boundary. Therefore,

$$\hat{H}_{||}^{i}\, e^{-j\beta_i \cdot r}\ a_y + \hat{H}_{||}^{r}\, e^{-j\beta_i \cdot r}\ a_y = \hat{H}_{||}^{t}\, e^{-j\beta_i \cdot r}\ a_y$$

There is no need to carry the $\mathbf{a_y}$ vector, because the magnetic fields only have one component in the **y** direction. Recall that this relation is valid at z = 0,

$$\hat{H}_{||}^{i}\, e^{-j\beta_i\,(\sin\theta_i x)} + \hat{H}_{||}^{r}\, e^{-j\beta_i\,(\sin\theta_r x)} = \hat{H}_{||}^{t}\, e^{-j\beta_i(\sin\theta_t x)} \qquad (6.13)$$

β_1 & β_{2_1} are the magnitudes of β in regions 1 & 2, respectively. In order for this to be valid at any value of **x** at any point on the interface, and knowing $\theta_i = \theta_r$:

$$\beta_1 \sin\theta_i = \beta_2 \sin\theta_t$$

Or

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{\beta_2}{\beta_1} = \frac{\dfrac{\omega}{V_2}}{\dfrac{\omega}{V_1}} = \frac{V_1}{V_2}$$

* This is the same relation that was determined earlier from Snell's Law. Substitute $\frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2}$ into equation 6.13 to obtain

$$\hat{H}_{||}^{i} + \hat{H}_{||}^{r} = \hat{H}_{||}^{t} \qquad \text{At } z = 0 \qquad (6.14)$$

E and H are related by η , so equation 6.14 can be rewritten as

$$\hat{E}_{||}^{i} + \hat{E}_{||}^{r} = \frac{\eta_1}{\eta_2}\hat{E}_{||}^{t} \qquad (6.15)$$

Tangential components of E must be continuous across the boundary, therefore

$$\hat{E}_{||}^{i} \cos\theta_i - \hat{E}_{||}^{r} \cos\theta_r = \hat{E}_{||}^{t} \cos\theta_t \quad \text{At } z = 0 \qquad (6.16)$$

*Remember the exponential terms cancel out z = 0, (Snell's Law).

Equations 6.15 & 6.16 are solved by –

$$\hat{E}_{||}^{r} = \hat{E}_{||}^{i} \frac{\eta_1 \cos\theta_i - \eta_2 \cos\theta_t}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t}$$

And

$$\hat{E}_{||}^{t} = \hat{E}_{||}^{i} \frac{2\eta_2 \cos\theta_i}{\eta_1 \cos\theta_i + \eta_2 \cos\theta_t} \qquad (6.17)$$

*Making use of the fact that $\theta_i = \theta_r$. Define the reflection coefficient $\hat{\Gamma}_{||}$ and the transmission $\hat{T}_{||}$:

$$\hat{\Gamma}_{\|} = \frac{\hat{E}_{\|}^{r}}{E_{\|}^{i}} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i} = \left. \frac{\cos\theta_t - \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i}{\cos\theta_t + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i} \right|_{\mu_1 = \mu_2 = \mu_\circ}$$

And

$$\hat{T}_{\|} = \frac{\hat{E}_{\|}^{t}}{E_{\|}^{i}} = \frac{2\eta_2 \cos\theta_t - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_t} = \left. \frac{2\cos\theta_i}{\cos\theta_i + \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\cos\theta_i} \right|_{\mu_1 = \mu_2 = \mu_\circ}$$

The total electric field in region 1 is

$$\hat{E}_{\|}^{tot} = \hat{E}_{\|}^{i} + \hat{E}_{\|}^{r} = \hat{E}_m^{i}(\cos\theta_i\, a_x - \sin\theta_i\, a_z)\, e^{-j\beta_i \cdot r} + \hat{E}_m^{r}(-\cos\theta_r\, a_x - \sin\theta_r\, a_z)\, e^{-j\beta_r \cdot r}$$

$$= \cos\theta_i\, \hat{E}_m^{i}\, e^{-j\beta x \sin\theta_i}\, (e^{-j\beta z \cos\theta_i} + \hat{\Gamma}_{\|} e^{j\beta z \cos\theta_i})\, a_x$$

$$+ \underbrace{\sin\theta_i\, \hat{E}_m^{i}\, e^{-j\beta x \sin\theta_i}}_{\text{Traveling – wave part}} \underbrace{\left(- e^{-j\beta z \cos\theta_i} + \hat{\Gamma}_{\|}\, e^{j\beta z \cos\theta_i}\right) a_z}_{\text{Standing plus traveling waves}} \qquad (6.18)$$

Substituted $\beta_i \cdot r, \beta_r \cdot r$ from expressions derived earlier, and $\hat{E}_m^{r} / \hat{E}_m^{i} = \hat{\Gamma}_{\|}$.
Equation 6.18 states that there is a traveling-wave field in the x direction, and a traveling and standing wave field in the z direction. The difference is that $\hat{\Gamma}_{\|} \neq 1$, but that $\hat{\Gamma}_{\|} = -\hat{E}_m^{r} / \hat{E}_m^{i}$. By rearranging the second term in a_x component of the total field –

$$\left[\left(1 - \hat{\Gamma}_{\|}\right) e^{-j\beta z \cos\theta_i} + 2\,\hat{\Gamma}_{\|}\left(\beta z \cos\theta_i\right)\right]$$

This expression indicates that a wave of amplitude $\left(1 - \hat{\Gamma}_{\|}\right)$ is propagating in the z direction and another wave of amplitude $\left(2\,\hat{\Gamma}_{\|}\right)$ has the characteristics of a standing wave along the z axis. The characteristic of the wave along the z axis is a combination of a traveling and standing wave. If $\hat{\Gamma}_{\|} = 1$ the amplitude of the traveling wave will be zero, and the wave characteristic along the z axis will be a totally standing wave. If $\hat{\Gamma}_{\|} = 0$, the amplitude of the standing wave will be zero and the wave characteristic in the z direction would be a totally traveling wave.

The magnetic field in region 1 is

$$\begin{aligned}\hat{H}_{\parallel}^{tot} &= \hat{H}_{\parallel}^{i} + \hat{H}_{\parallel}^{r} = \hat{H}_{m}^{i}\, e^{-j\beta_i \cdot r}\, a_y \;\; + \hat{H}_{m}^{r}\, e^{-j\beta_r \cdot r}\, a_y \\ &= \frac{\hat{E}_m^i}{\eta_1} e^{-j\beta\, x\sin\theta_i} \left(e^{-j\beta\, z\cos\theta_i} + \frac{\hat{E}_m^i}{\hat{E}_m^r} e^{-j\beta\, x\cos\theta_i} \right) a_y \\ &= \frac{\hat{E}_m^i}{\eta_1} e^{-j\beta x\sin\theta_i} \left(e^{-j\beta\, z\cos\theta_i} - \hat{\Gamma}_{\parallel}\, e^{j\beta z\cos\theta_i} \right) a_y\end{aligned}$$

The transmitted fields in medium 2 are

$$\begin{aligned}\hat{E}_{\parallel}^{i} &= \hat{E}_m^t \left(\cos\theta_t\, a_x - \sin\theta_t\, a_z\right) e^{-j\beta_t \cdot r} \\ &= \hat{T}_{\parallel}\, \hat{E}_m^i \left(\cos\theta_t\, a_x - \sin\theta_t\, a_z\right) e^{-j\beta_t \cdot r}\end{aligned}$$

And

$$\hat{H}_{\parallel}^{t} = \hat{H}_m^t\, a_y\, e^{-j\beta_t \cdot r} = \frac{\hat{T}_{\parallel}\, \hat{E}_m^i}{\eta_2} e^{-j\beta_t \cdot r}\, a_y$$

Where $\beta_t \cdot r = \beta_2 \left(x\sin\theta_t + z\cos\theta_t\right)$ and $\hat{E}_m^t / \hat{E}_m^i = T_{\parallel}$.

<u>Definition:</u>

Brewster Angle – (from Brewster's Law), the polarizing angle of which (when light is incident) the reflected and refracted index is equal to the tangent of the polarizing angle. In other words, the angle of incidence of which there is no reflection.
From the reflection coefficient expression-

$$\hat{\Gamma}_{\parallel} = \frac{\eta_2 \cos\theta_t - \eta_1 \cos\theta_i}{\eta_2 \cos\theta_t + \eta_1 \cos\theta_i}$$

It can be seen that there is an angle of incidence at $\hat{\Gamma}_{||} = 0$. This angle can be obtained when

$$\eta_1 \cos\theta_i = \eta_2 \cos\theta_t$$

Or

$$\cos\theta_i = \frac{\eta_2}{\eta_1}\cos\theta_t \qquad (6.19)$$

The angle of incidence θ_i , at which $\hat{\Gamma}_{||} = 0$, is known as the Brewster angle. The expression for this angle in terms of the dielectric properties of media 1 & 2, considering Snell's Law for the special case $\mu_1 = \mu_2 = \mu_\circ$ is

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{V_1}{V_2}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\Bigg|_{\mu_1=\mu_2=\mu_\circ}$$

This condition is important, because it is usually satisfied by the materials often used in optical applications.

Equation 6.19 will take the form –

$$\cos\theta_i = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}}\cos\theta_t \qquad (6.20)$$

Square both sides of equation 6.20 and use Snell's Law for the special case of $\mu_1 = \mu_2 = \mu_\circ$ for the following result:

$$\cos^2\theta_i \frac{\varepsilon_1}{\varepsilon_2} = \cos^2\theta_t = \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_t\right)$$

$$= \frac{\varepsilon_1}{\varepsilon_2}\left(1 - \sin^2\theta_i\right)$$

The last substitution was based on Snell's Law of refraction. Therefore,

$$\left(1 - \sin^2\theta_i\right) = \frac{\varepsilon_1}{\varepsilon_2} - \frac{\varepsilon_1^2}{\varepsilon_2^2}\sin^2\theta_i$$

$$1-\frac{\varepsilon_1}{\varepsilon_2}=\sin^2\theta_i\left(1-\frac{\varepsilon_1^2}{\varepsilon_2^2}\right)$$

And

$$\sin^2\theta_i=\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1} \tag{6.21}$$

The Brewster angle of incidence is

$$\sin\ \theta_i\ =\sqrt{\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}} \tag{6.22}$$

A specific value of θ_i can be obtained from equation 6.21 -

$$1-\cos^2\theta_i=\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}$$

Or

$$\cos^2\theta_i=1-\frac{\varepsilon_2}{\varepsilon_2+\varepsilon_1}=\frac{\varepsilon_1}{\varepsilon_2+\varepsilon_1}=$$

$$\cos\ \theta_i=\sqrt{\frac{\varepsilon_1}{\varepsilon_2+\varepsilon_1}} \tag{6.23}$$

From equations 6.22 & 6.23 –

$$\tan\theta_i=\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

This specific angle of incidence θ_i is called the Brewster angle θ_β.

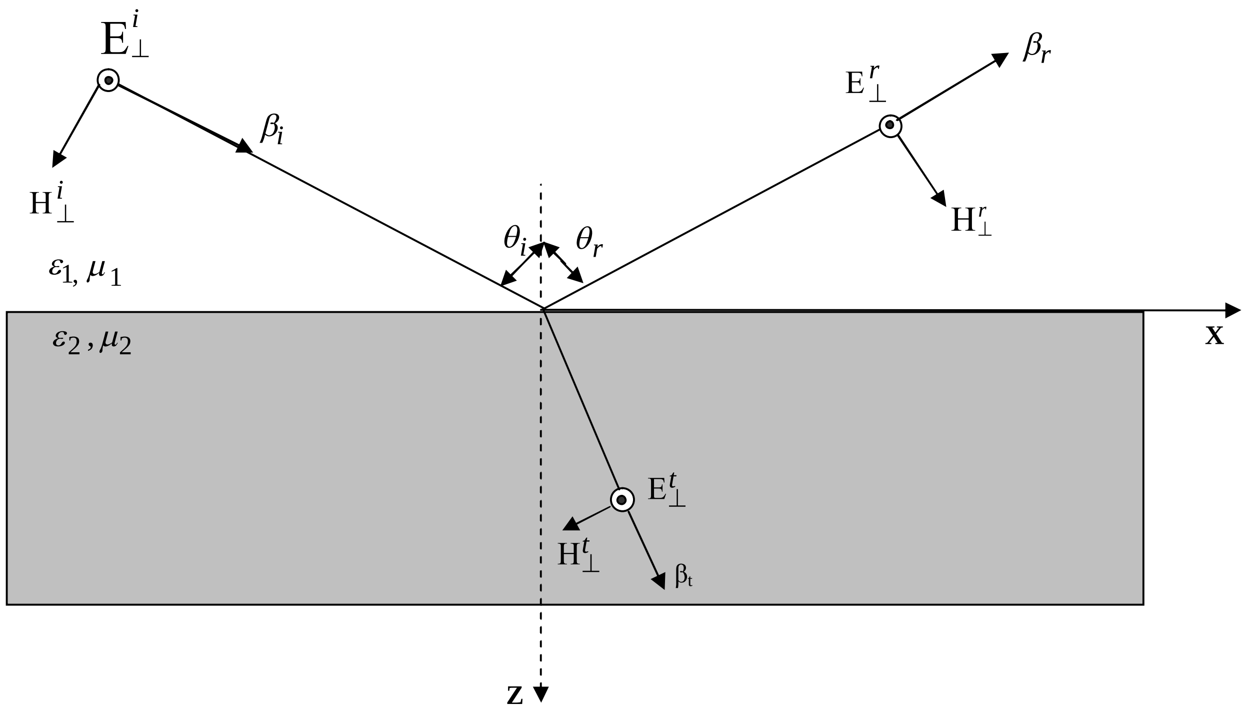
$$\theta_\beta=\tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Perpendicular Polarization case – E Normal to Plane of Incidence

As shown in figure 6.10 is a perpendicular polarized wave incident at angle θ_i a dielectric medium 2. Snell's Law states that a reflected wave will be at the same angle $\theta_r = \theta_i$, and the transmitted wave in medium 2 at angle θ_t can be calculated using this law. The amplitude of the reflected and transmitted waves can be determined by applying the continuity of the tangential components of E & H at the boundary.

This is given by –

$$\hat{H}_{\perp}^{i} \cos\theta_i - \hat{H}_{\perp}^{r} \cos\theta_i = \hat{H}_{\perp}^{t} \cos\theta_t$$



Since E & H are related by η,

$$\frac{\hat{E}_{\perp}^{i}}{\eta_1}\cos\theta_i - \frac{\hat{E}_{\perp}^{r}}{\eta_1}\cos\theta_i = \frac{\hat{E}_{\perp}^{t}}{\eta_2}\cos\theta_t \qquad (6.24)$$

$$\hat{E}_{\perp}^{i} + \hat{E}_{\perp}^{r} = \hat{E}_{\perp}^{t} \qquad \text{At } z = 0 \qquad (6.25)$$

*Note: The exponential factors were canceled after substituting z = 0 and using Snell's Laws in the above two equations.

$$\hat{\Gamma}_{\perp} = \frac{\hat{E}_{\perp}^{r}}{\hat{E}_{\perp}^{i}} = \frac{\eta_2 \cos\theta_i - \eta_1 \cos\theta_t}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

And for nonmagnetic materials, $\mu_1 = \mu_2 = \mu_\circ$,

$$\hat{\Gamma}_{\perp} = \frac{\cos\theta_t - \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\cos\theta_t}{\cos\theta_t + \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\cos\theta_t}$$

at z = 0,

$$\hat{T}_{\perp} = \frac{\hat{E}_{\perp}^{t}}{\hat{E}_{\perp}^{i}} = \frac{2\eta_2 \cos\theta_i}{\eta_2 \cos\theta_i + \eta_1 \cos\theta_t}$$

For nonmagnetic material,

$$\hat{T}_{\perp} = \frac{2\cos\theta_i}{\cos\theta_i \sqrt{\dfrac{\varepsilon_2}{\varepsilon_1}}\cos\theta_t}$$

Comparison between Reflection Coefficients $\hat{\Gamma}_{\parallel}$ and $\hat{\Gamma}_{\perp}$ for Parallel and Perpendicular Polarizations

The significant differences between the two will be illustrated in the following example:

<u>EXAMPLE</u>

1. Define what is meant by the Brewster angle.
2. Calculate the polarization angle (Brewster angle) for an air water$\left(\varepsilon_r = 81\right)$ interface at which plane waves pass from the following:
   (a) Air into water.
   (b) Water into air.

SOLUTION

1. *Brewster angle* is defined as the angle of incidence at which there will be no reflected wave. It occurs when the incident wave is polarized such that the **E** field is parallel to the plane of incidence.
2. (a) Air into water:

$$\varepsilon_{r1} = 1 \quad and \quad \varepsilon_{r2} = 81$$

The Brewster angle is then given by

$$\theta_\beta = \tan^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = 6.34°$$

Therefore,

$$\theta_\beta = \tan^{-1}\sqrt{81} = 83.7°$$

(b) Water into air:

$$\varepsilon_{r1} = 81 \quad and \quad \varepsilon_{r2} = 1$$

Hence,

$$\theta_\beta = \tan^{-1}\sqrt{\frac{1}{81}} = 6.34°$$

To relate the Brewster angles in both cases, let us calculate the angle of refraction.

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

Therefore, in case a,

$$\frac{\sin\theta_B}{\sin\theta_t} = \sqrt{81}$$

Therefore,

$$\sin\theta_t = \frac{\sin 83.7}{9} = 0.11$$

Or $\theta_t = 6.34^\circ$, which is the same as the Brewster angle for case b. Also, the angle of refraction in case b is given by Snell's Law as:

$$\frac{\sin\theta_B}{\sin\theta_t} = \sqrt{\frac{\varepsilon_\circ}{81\varepsilon_\circ}} = \sqrt{\frac{1}{81}}$$

Therefore,

$$\sin\theta_t = \frac{\sin 6.34^\circ}{\sqrt{\frac{1}{81}}} = 0.99$$

Or $\theta_t = 83.7^\circ$, which is the Brewster angle for case a.

Total Reflection at Critical Angle of Incidence

In the previous section it was shown that for common dielectrics, the phenomenon of total transmission exists only where the electric field is parallel to the plane of incidence known as parallel polarization.
There is a second phenomenon existing for both polarizations:

- Total reflection occurring at the interface between two dielectric media
- A wave passing from a medium with a larger dielectric constant to a medium with smaller value of ε

Snell's Law of refraction shows –

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} \qquad \text{or} \qquad \sin\theta_i = \frac{\sin\theta_t}{\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}} \tag{6.26}$$

Therefore, if $\varepsilon_1 > \varepsilon_2, and\ \theta_t > \theta_i$ then a wave incident at an angle θ_i will pass into medium 2 at a larger angle θ_t.

Definition:

θ_c, (critical angle of incidence) is the value of θ_i that makes $\theta_t = \pi/2$, see Figure 6.13.

Substitute $\theta_t = \pi/2$ in equation 6.26 to get –

$$\sin\theta_c = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}, \text{ or } \theta_c = \sin^{-1}\sqrt{\frac{\varepsilon_2}{\varepsilon_1}}$$

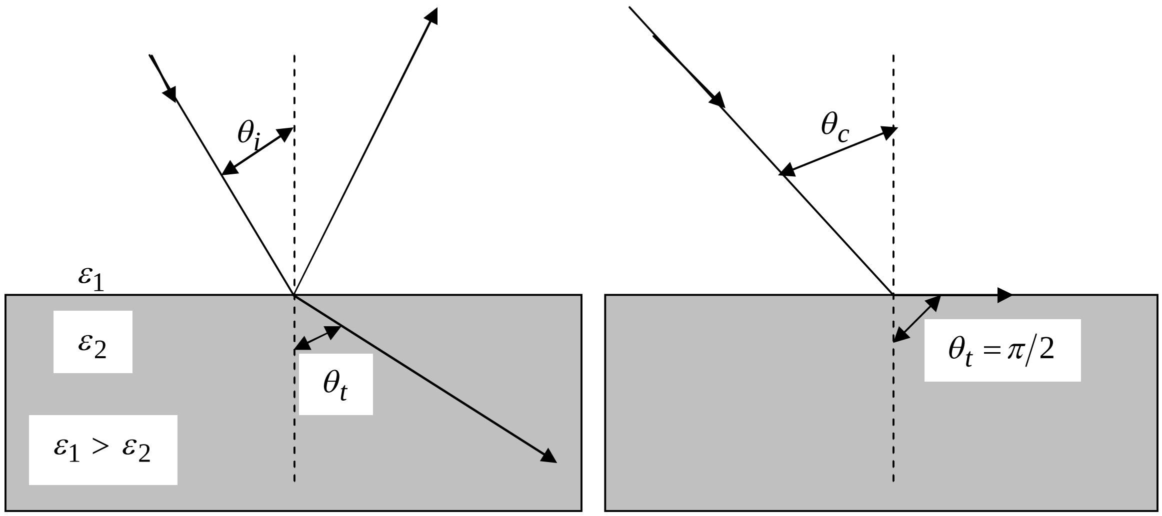
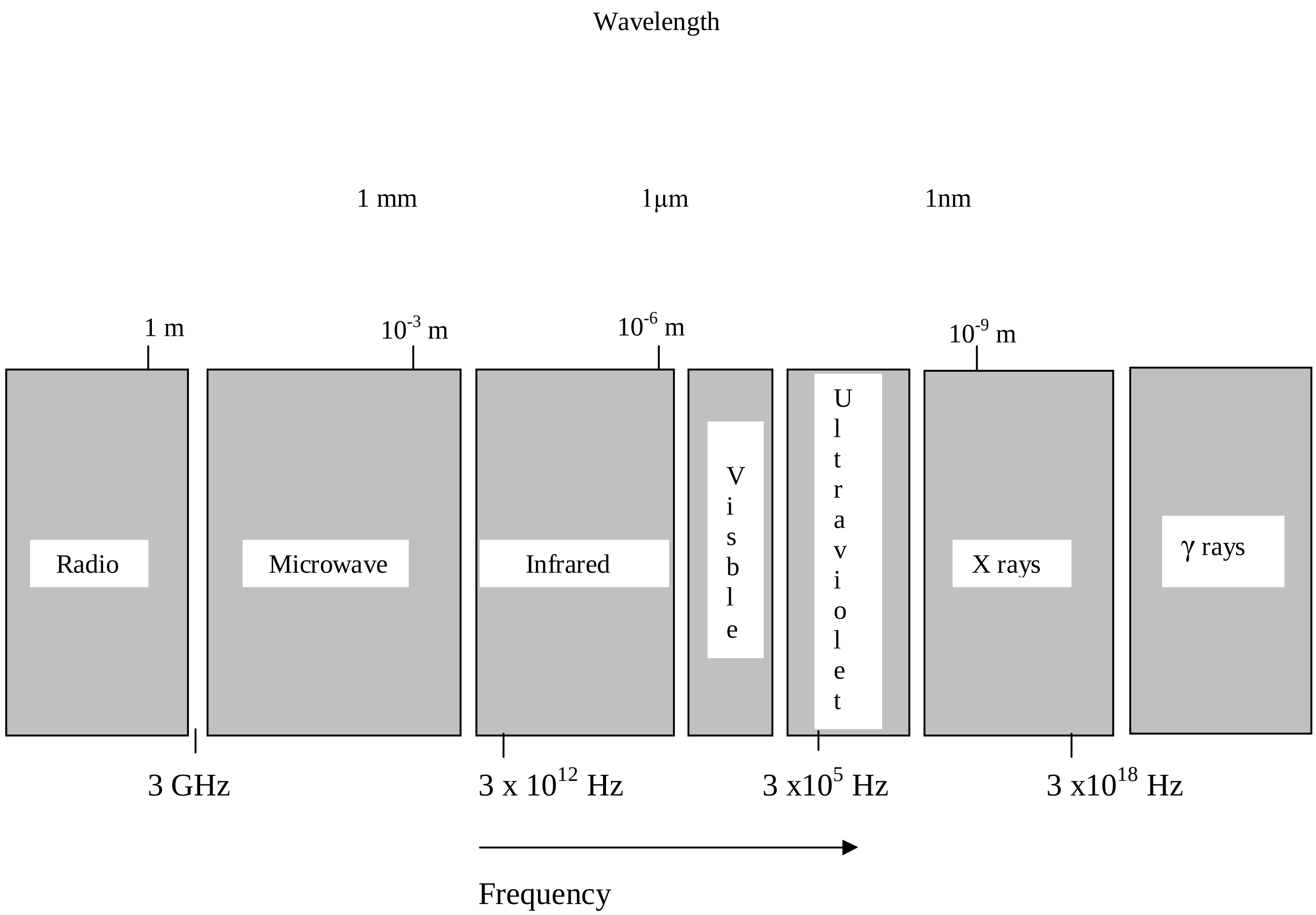


Figure illustrates the fact that $\theta_t > \theta_i$, *if* $\varepsilon_1 > \varepsilon_2$. The critical angle θ_c is defined as the value of θ_i at which $\theta_t = \pi/2$.

Envision a beam of light impinging on an interface between two transparent media where $n_i < n_t$. At normal incidence ($\theta_i = 0$) most of the incoming light is transmitted into the less dense medium. As θ_i increases, more and more light is reflected back into the dense medium, while θ_t increases. When $\theta_t = 90°$, θ_i is defined to be θ_c and the transmittance becomes zero. For $\theta_i > \theta_c$ all of the light is totally internally reflected, remaining in the incident medium.

Electromagnetic Spectrum

Figure 6.16 Electromagnetic spectrum from radio waves to X and γ rays.



Application to Optics

The figure above shows the spectrum of electromagnetic radiation extending from the long- wavelength radio waves to X rays and gamma rays the shortest wavelength.

Topics to be discussed will include control of polarization of incident waves, role of Brewster windows in light amplification, and use of the concept of angle of total reflection in optical fibers.

Polarization by Reflection

Definition

Unpolarized light – light in which the wave orientation is random around the axis of the beam.

Unpolarized light has both polarization cases

- Parallel polarization, where the electric field is the plane of incidence
- Perpendicular polarization where the electric field is perpendicular to the plane of incidence

In certain cases, there may be a need to separate the two polarizations. One method that can be used is the Brewster angle of incidence, also called the polarization angle, to separate the two orthogonal polarizations.

Example

Consider an Unpolarized light that is incident at the Brewster angle on a piece of glass with index of refraction $n = \sqrt{\varepsilon_r} = 1.5$. The polarization with a electric field parallel to the plane of incidence will be entirely transmitted and the other polarization with a electric field perpendicular to the plane of incidence will be partially reflected and partially transmitted. Why is the electric field parallel to the plane of incident totally transmitted? *Because it is incident at the Brewster angle.

The second interface which is glass to air as illustrated in example 6.7 has an angle of incidence also known as the Brewster angle for light incident from the glass side to free space. So, again the polarization with **E** parallel to the plane of incident will be entirely transmitted, and **E** perpendicular will be partially reflected and partially transmitted.

In Figure 6.17:

- Reflected wave is entirely polarized, **E** perpendicular to the plane of incidence
- Transmitted wave possess both polarizations
- Larger amplitude is the **E** parallel to plane of incidence – entirely transmitted throughout the interfaces
- More glass elements and the transmitted light could be essentially completely polarized, **E** parallel to the plane of incidence

Brewster Windows or Brewster Cuts in LASER

In a normal situation there are more electrons in the ground state (level 1) than in the excited states (level 2 & 3). In other words, there are more electrons in level 1 ready to absorb photons that there are electrons in level 2 & 3 to emit photons. A net emission of photons could be the result if this situation could be inverted. Such a condition is called population inversion. This in fact is the fundamental principle involved in the operation of a laser. Figure 6.8 illustrates this principle.

Definition:

Laser (Light Amplification by Stimulated Emission) – A device that produces coherent radiation in the visible-light range, between 7500 and 3900 angstroms

Summarized steps leading to LASER action in three-level ruby laser material:

1. The laser material is in the shape of a long rod that is subjected to radiation from an extremely intense light source that causes interatomic transition from energy levels 1 to 3. (Figure 6.18b)

2. If the nonradiative transition between level 3 and level 2 is fast enough, then electrons in level 3 will transfer to level instead of returning to level 1.

3. As a result of direct transition the population of electrons in level 2 will increase from level 1. This is during the radiation from the light source, as well as the transfer from level 3. (Figure 6.18c)

4. If the pumping action is large and fast enough the electron population at level 2 can be made larger than level 1. Radiation of light quanta at frequency f_{21} occurs when the electrons can make the transition from level 2 to level 1.

5. By placing mirrors at the end of the laser and forcing the radiation to be reflected back and forth maintaining the high-photon density, stimulated emission will increase resulting in a large photon density build up or in other words an avalanche of photons.

6. An intense light beam will result emerging from the end of the laser rod.

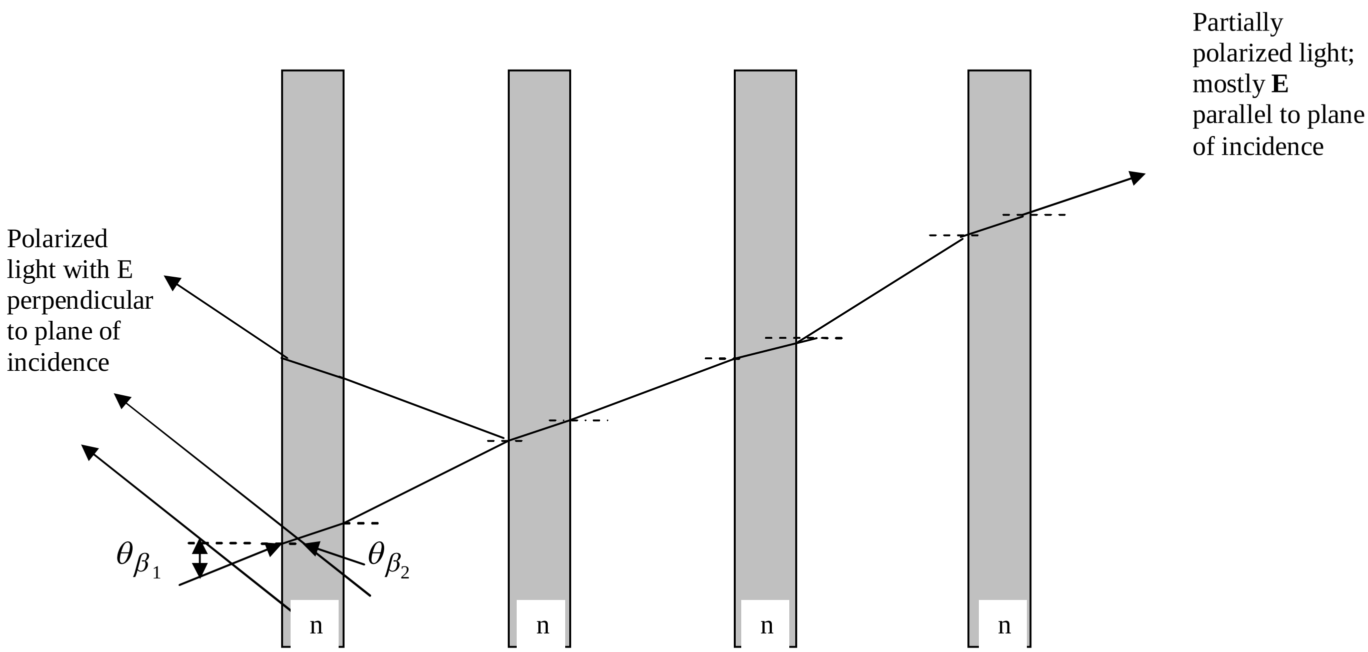


Figure :Light polarizations by multiple reflections.

Figure 6.18 is a schematic diagram illustrating the sequence of events.
The role of the Brewster angle:

<u>Known Factors</u>

- The output of many lasers is linearly polarized
- The ratio of the light polarized in one direction exceeds the light polarized in the orthogonal direction by 1000:1

As in most cases, a high degree of linear polarization will be the result of a Brewster surface within the laser. A Brewster surface is usually used in the construction of a laser. The light must be transmitted out of the medium of the laser to avoid minimal loss.

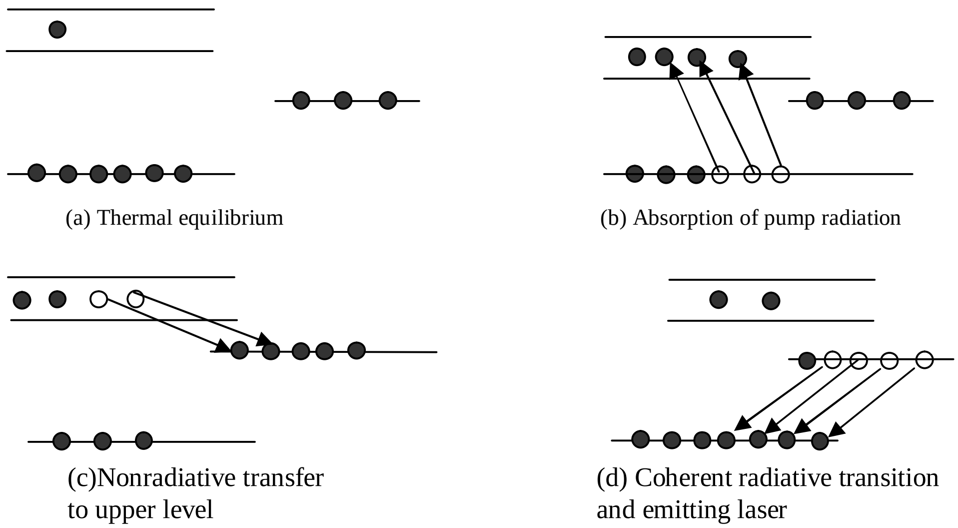
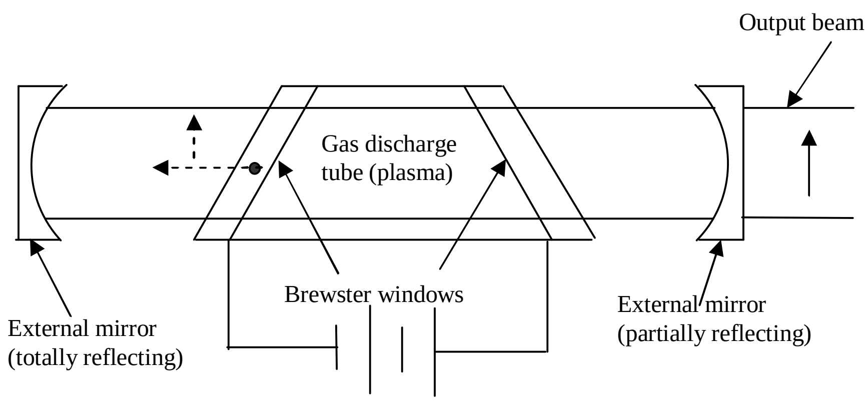


Figure :Sequence of events occurring in laser action.

Figure is a schematic illustrating the use of Brewster windows in a gas discharge laser. The Brewster angle makes sure that light in one polarization direction is transmitted out of the medium of the laser to the reflecting mirrors and back into the medium of the laser with no loss. Where the light is polarized perpendicular to the plane of incidence a large loss at the Brewster surface will take place due to the reflection out of the medium of the laser. The preferred polarization case (linear polarization) will lase (emit coherent light) that will account for the high degree of polarization taking place at the output.

Figure : Schematic illustrating the use of Brewster windows in a gas discharge LASER



The device in Figure 6.19 exhibits stimulated emission of radiation. For and example lets say the mixture of gases are helium and neon. These gases are confined to the glass tube sealed at both ends by mirrors. An oscillator is connected to the tube to that causes electrons to sweep through the tube, colliding with atoms of gas and raising them to exited states. Some neon atoms are excited to a higher state during this process that will also result in a collision with excited helium atoms. Stimulated emission occurs as the neon atoms make a transition to a lower state and neighboring excited atoms are stimulated to emit at the same frequency and phase. This will result in a production of coherent light.

Fiber Optics

Fiber optics deals with the transmission of light through small filamentary fibers called dielectric waveguides. This is based on the phenomenon of total internal reflection occurring at the point where the light is obliquely incident on an interface between two media with different refractive indexes at an angle greater than the critical angle. Light is incident at an angle θ_i as shown in Figure 6.20 and is required to determine the range of values of the index of refraction **n** so the internal reflections will occur for any value of θ_i.

Snell's Law of refraction is the relationship between θ_i and θ_t as the wave enter the fiber is

$$\frac{\sin\theta_i}{\sin\theta_t} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}} = n \qquad \varepsilon_1 = \varepsilon_\circ \tag{6.27}$$

If θ_2 is suppose to be larger than θ_c, then

$$\sin\,\theta_2 = \cos\,\theta_t \geq \sin\,\theta_c \tag{6.28}$$

Refraction from fiber to air sin θ_c = 1/n, therefore, from equation 6.27 & 6.28 –

$$\sin\theta_2 = \cos\theta_t = \sqrt{1-\sin^2\theta_t} = \sqrt{1-\left(\frac{1}{n}\right)^2 \sin^2\theta_i} \geq \frac{1}{n} \tag{6.29}$$

Solve for n,

$$n^2 \geq\ 1+\sin^2\,\theta_i \tag{6.30}$$

For equation 6.30 to be true then $= \pi/2$, all incident light will be passed by the fiber requiring

$$n^2 \geq 2 \quad \text{or} \quad n \geq \sqrt{2}$$

Most types of glass have n ≈ 1.5; therefore, we have a valid equation.

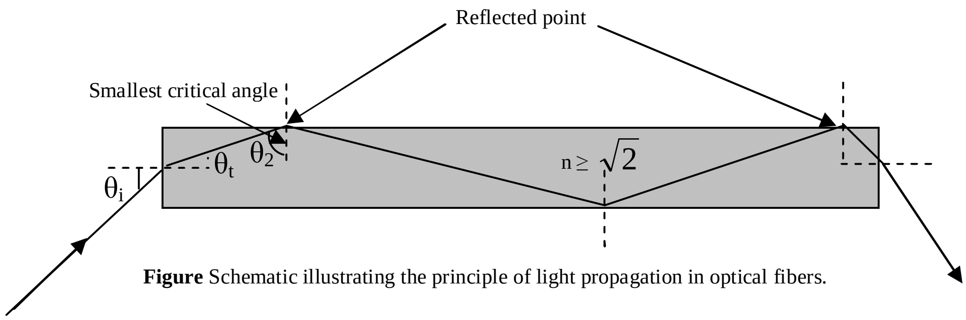
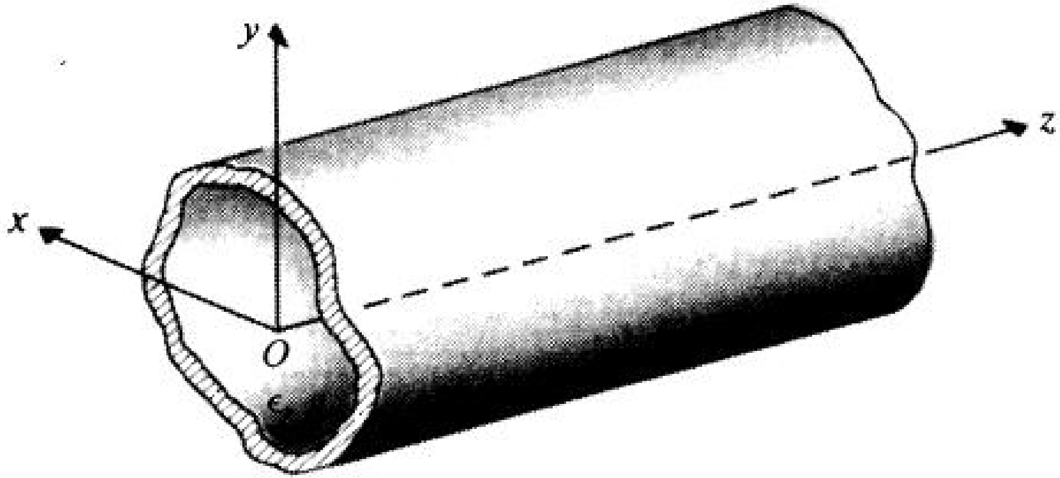


Figure Schematic illustrating the principle of light propagation in optical fibers.

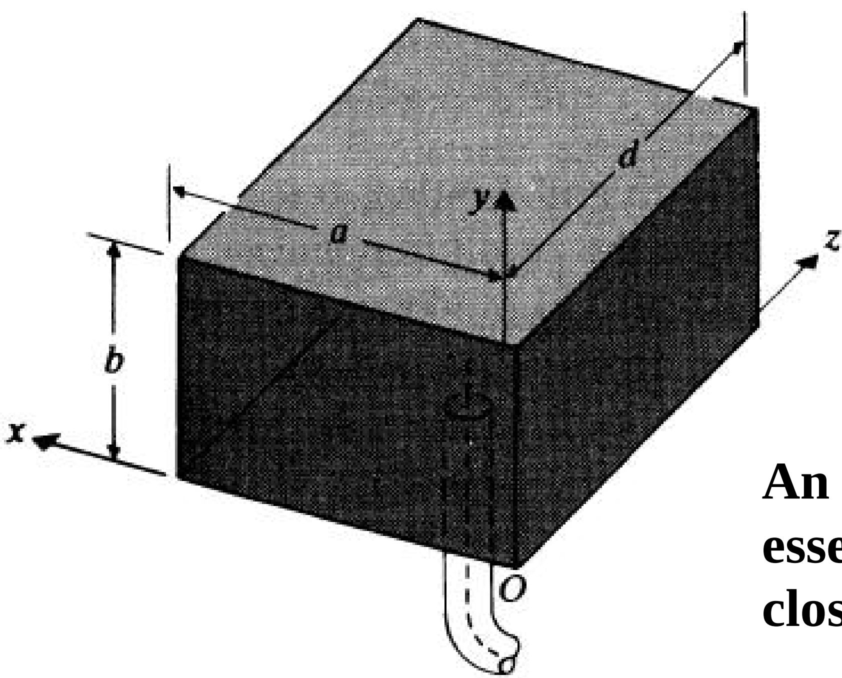
UNIT-5

WAVEGUIDES AND CAVITY RESONATORS

TEM wave: Transverse electromagnetic wave (no field components in the direction of propagation)
TM wave: Transverse magnetic wave (with a longitudinal electric field component in the direction of propagation)
TE wave: Transverse electric wave (with a longitudinal magnetic field component in the direction of propagation)



A uniform waveguide with an arbitrary cross section



An enclosed conducting box, which is essentially a segment of a waveguide with closed end faces, is called a cavity resonator

Waveguides

Transmission Lines: support transverse electromagnetic (TEM) waves.

- Waveguide: A structure in which EM waves can propagate
- TLs are specials cases of waveguides
- Waveguides support TEM waves ($H_z=E_z=0$), but also support transverse magnetic (TM) waves ($H_z=0$) and transverse electric (TE) waves ($E_z=0$)
- TM and TE modes have characteristic *cut-off frequencies*
- Waveguides can be metallic or dielectric

.

General Wave Behaviors along Uniform Guiding Structures

We shall consider a straight waveguide in the form of a dielectric-filled metal tube having an arbitrary cross section and lying along the z-axis.

Helmholtz's equations:

$$\nabla^2 \mathbf{E} + k^2 \mathbf{E} = 0 \tag{10–3}$$

and

$$\nabla^2 \mathbf{H} + k^2 \mathbf{H} = 0, \tag{10–4}$$

where **E** and **H** are three-dimensional vector phasors, and k is the wavenumber:

$$k = \omega\sqrt{\mu\epsilon}. \tag{10–5}$$

The three-dimensional Laplacian operator ∇^2 may be broken into two parts: $\nabla^2_{u_1 u_2}$ for the cross-sectional coordinates and ∇^2_z for the longitudinal coordinate. For waveguides with a rectangular cross section we use Cartesian coordinates:

$$\begin{aligned} \nabla^2 \mathbf{E} &= (\nabla^2_{xy} + \nabla^2_z)\mathbf{E} = \left(\nabla^2_{xy} + \frac{\partial^2}{\partial z^2}\right)\mathbf{E} \\ &= \nabla^2_{xy}\mathbf{E} + \gamma^2 \mathbf{E}. \end{aligned} \tag{10–6}$$

Combination of Eqs. (10–3) and (10–6) gives

$$\nabla^2_{xy}\mathbf{E} + (\gamma^2 + k^2)\mathbf{E} = 0. \tag{10–7}$$

γ is the propagation constant

Similarly, from Eq. (10–4) we have

$$\nabla_{xy}^2 \mathbf{H} + (\gamma^2 + k^2)\mathbf{H} = 0. \tag{10–8}$$

In Maxwell equations, the two source-free curl equations have six components:

From $\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$:		From $\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E}$:	
$\frac{\partial E_z^0}{\partial y} + \gamma E_y^0 = -j\omega\mu H_x^0$	(10–9a)	$\frac{\partial H_z^0}{\partial y} + \gamma H_y^0 = j\omega\epsilon E_x^0$	(10–10a)
$-\gamma E_x^0 - \frac{\partial E_z^0}{\partial x} = -j\omega\mu H_y^0$	(10–9b)	$-\gamma H_x^0 - \frac{\partial H_z^0}{\partial x} = j\omega\epsilon E_y^0$	(10–10b)
$\frac{\partial E_y^0}{\partial x} - \frac{\partial E_x^0}{\partial y} = -j\omega\mu H_z^0$	(10–9c)	$\frac{\partial H_y^0}{\partial x} - \frac{\partial H_x^0}{\partial y} = j\omega\epsilon E_z^0$	(10–10c)

Since divE=0 and divH=0 (E and H are divergenceless), this means that there are only four independent equations in the above table. Therefore, we have……

Four independent equations:

$$H_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial x} - j\omega\epsilon \frac{\partial E_z^0}{\partial y}\right),$$

$$H_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial H_z^0}{\partial y} + j\omega\epsilon \frac{\partial E_z^0}{\partial x}\right),$$

$$E_x^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial x} + j\omega\mu \frac{\partial H_z^0}{\partial y}\right),$$

$$E_y^0 = -\frac{1}{h^2}\left(\gamma \frac{\partial E_z^0}{\partial y} - j\omega\mu \frac{\partial H_z^0}{\partial x}\right)$$

The transverse components of E and H can be expressed in terms of the longitudinal field components Ez, Hz in the direction of propagation (z-direction).

$$h^2 = \gamma^2 + k^2$$ (h is viewed as a transverse wavenumber)

Discussions: Single-conductor waveguides cannot support the TEM waves

1) If h is nonzero, and Ez=0, Hz=0, then all the transverse components of E and H vanish. This means that TEM cannot exist in this single-conductor waveguide.

2) If h is zero, single-conductor waveguides cannot support TEM waves, either .

According to whether Ez or Hz exists, the propagating waves in a uniform waveguide can be classified into three types:

1. *Transverse electromagnetic (TEM) waves.* These are waves that contain neither E_z nor H_z. We encountered TEM waves in Chapter 8 when we discussed plane waves and in Chapter 9 on waves along transmission lines.
2. *Transverse magnetic (TM) waves.* These are waves that contain a nonzero E_z but $H_z = 0$.
3. *Transverse electric (TE) waves.* These are waves that contain a nonzero H_z but $E_z = 0$.

TEM waves

(For TEM waves, Ez=0, Hz=0, and hence h=0, i.e., γ=jk)

.

$$u_{p(\mathrm{TEM})} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\epsilon}} \qquad \text{(m/s)}.$$

$$\gamma_{\mathrm{TEM}} = jk = j\omega\sqrt{\mu\epsilon}, \tag{10–17}$$

which is exactly the same expression for the propagation constant of a uniform plane wave in an unbounded medium characterized by constitutive parameters ϵ and μ.

We can obtain the ratio between E_x^0 and H_y^0 from Eqs. (10–9b) and (10–10a) by setting E_z and H_z to zero. This ratio is called the ***wave impedance***. We have

$$Z_{\text{TEM}} = \frac{E_x^0}{H_y^0} = \frac{j\omega\mu}{\gamma_{\text{TEM}}} = \frac{\gamma_{\text{TEM}}}{j\omega\epsilon}, \tag{10–19}$$

$$Z_{\text{TEM}} = \sqrt{\frac{\mu}{\epsilon}} = \eta \qquad (\Omega)$$

a TEM wave propagating in the $+z$-direction:

$$\boxed{\mathbf{H} = \frac{1}{Z_{\text{TEM}}}\mathbf{a}_z \times \mathbf{E} \qquad (\text{A/m}),}$$

For TEM waves, k, E, H form a right-handed system.

TM waves

(Hz=0, and all the transverse components of E, H are expressed in terms of Ez)

E, H are expressed in terms of Ez:

$$H_x^0 = \frac{j\omega\epsilon}{h^2}\frac{\partial E_z^0}{\partial y},$$

$$H_y^0 = -\frac{j\omega\epsilon}{h^2}\frac{\partial E_z^0}{\partial x},$$

$$E_x^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial x},$$

$$E_y^0 = -\frac{\gamma}{h^2}\frac{\partial E_z^0}{\partial y}.$$

Impedance Z of TM waves is equal to $\gamma/(j\omega\varepsilon)$ [similar to that of TEM waves], but not equal to $j\omega\,\mu / \gamma$.

$$Z_{TM} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{\gamma}{j\omega\epsilon} \qquad (\Omega).$$

$$\mathbf{H} = \frac{1}{Z_{TM}}(\mathbf{a}_z \times \mathbf{E}) \qquad \text{(A/m)}.$$

The relation between E and H of TM waves is of the same form as that of TEM waves.

Cutoff frequency of TM waves (when γ=0)

$$\begin{aligned} \gamma &= \sqrt{h^2 - k^2} \\ &= \sqrt{h^2 - \omega^2\mu\epsilon}. \end{aligned} \tag{10–33}$$

Two distinct ranges of the values for the propagation constant are noted, the dividing point being $\gamma = 0$, where

$$\omega_c^2\mu\epsilon = h^2 \tag{10–34}$$

or

$$\boxed{f_c = \frac{h}{2\pi\sqrt{\mu\epsilon}} \qquad \text{(Hz)}.} \tag{10–35}$$

The frequency, f_c, at which $\gamma = 0$ is called a ***cutoff frequency***. *The value of f_c for a particular mode in a waveguide depends on the eigenvalue of this mode.* Using Eq. (10–35), we can write Eq. (10–33) as

$$\gamma = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2}. \tag{10–36}$$

When γ=0, the frequency contributes only to the transverse wavenumber h. Such a frequency value is the cutoff frequency.

a) $\left(\frac{f}{f_c}\right)^2 > 1$, or $f > f_c$. In this range, $\omega^2\mu\epsilon > h^2$ and γ is imaginary. We have, from Eq. (10–33),

$$\gamma = j\beta = jk\sqrt{1-\left(\frac{h}{k}\right)^2} = jk\sqrt{1-\left(\frac{f_c}{f}\right)^2}. \tag{10–37}$$

The phase velocity of the propagating wave in the guide is

$$u_p = \frac{\omega}{\beta} = \frac{u}{\sqrt{1-(f_c/f)^2}} = \frac{\lambda_g}{\lambda}u > u.$$

$$u_g = \frac{1}{d\beta/d\omega} = u\sqrt{1-\left(\frac{f_c}{f}\right)^2} = \frac{\lambda}{\lambda_g}u < u.$$

Thus,

$$u_g u_p = u^2.$$

This relation is exactly the same as that of a de Broglie matter wave (in relativistic quantum mechanics). Indeed, the dispersion relation of the wave in a waveguide is exactly analogous to that of a de Broglie matter wave (with rest mass proportional to the transverse wavenumber h).

The wave impedance Z is a real number (purely resistive).

$$Z_{\mathrm{TM}} = \eta \sqrt{1 - \left(\frac{f_c}{f}\right)^2} \qquad (\Omega).$$

b) $\left(\frac{f}{f_c}\right)^2 < 1$, or $f < f_c$. When the operating frequency is lower than the cutoff frequency, γ is real and Eq. (10–36) can be written as

$$\gamma = \alpha = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2}, \qquad f < f_c, \tag{10–46}$$

which is, in fact, an attenuation constant. Since all field components contain the propagation factor $e^{-\gamma z} = e^{-\alpha z}$, the wave diminishes rapidly with z and is said to be ***evanescent***. Therefore, ***a waveguide exhibits the property of a high-pass filter***. *For a given mode, only waves with a frequency higher than the cutoff frequency of the mode can propagate in the guide.*

Substitution of Eq. (10–46) in Eq. (10–31) gives the wave impedance of TM modes for $f < f_c$:

$$Z_{\mathrm{TM}} = -j\frac{h}{\omega\epsilon}\sqrt{1 - \left(\frac{f}{f_c}\right)^2}, \qquad f < f_c. \tag{10–47}$$

Thus, the wave impedance of evanescent TM modes at frequencies below cutoff is purely reactive, indicating that there is no power flow associated with evanescent waves. **The wave impedance Z is reactive (no power flow with evanescent waves)**

TE waves

(Ez=0, and all the transverse components of E, H are expressed in terms of Hz)

E, H are expressed in terms of Hz:

$$H_x^0 = -\frac{\gamma}{h^2}\frac{\partial H_z^0}{\partial x},$$

$$H_y^0 = -\frac{\gamma}{h^2}\frac{\partial H_z^0}{\partial y},$$

$$\mathbf{E} = -Z_{TE}(\mathbf{a}_z \times \mathbf{H}) \qquad (\text{V/m}).$$

$$E_x^0 = -\frac{j\omega\mu}{h^2}\frac{\partial H_z^0}{\partial y},$$

$$E_y^0 = \frac{j\omega\mu}{h^2}\frac{\partial H_z^0}{\partial x}.$$

Impedance Z of TM waves is equal to jωμ/γ[similar to that of TEM waves], but not equal to γ/(jωε) .

$$\boxed{Z_{TE} = \frac{E_x^0}{H_y^0} = -\frac{E_y^0}{H_x^0} = \frac{j\omega\mu}{\gamma} \qquad (\Omega).}$$

When the frequency f is larger than fc,

a) $\left(\frac{f}{f_c}\right)^2 > 1$, or $f > f_c$. In this range, γ is imaginary, and we have a propagating mode. The expression for γ is the same as that given in Eq. (10–37):

$$\gamma = j\beta = jk\sqrt{1 - \left(\frac{f_c}{f}\right)^2}. \tag{10–56}$$

Consequently, the formulas for β, λ_g, u_p, and u_g in Eqs. (10–38), (10–39), (10–42), and (10–43), respectively, also hold for TE waves. Using Eq. (10–56) in Eq. (10–54), we obtain

$$\boxed{Z_{TE} = \frac{\eta}{\sqrt{1 - (f_c/f)^2}} \qquad (\Omega),} \tag{10–57}$$

Note:
1) Wavelength, phase velocity, group velocity are the same for TE and TM waves.
2) The wave impedance Z of TE waves is larger than the dielectric impedance, while the wave impedance Z of TM waves is smaller than the dielectric impedance.

When the frequency f is smaller than fc,

b) $\left(\frac{f}{f_c}\right)^2 < 1$, or $f < f_c$. In this case, γ is real and we have an evanescent or non-propagating mode:

$$\gamma = \alpha = h\sqrt{1 - \left(\frac{f}{f_c}\right)^2}, \qquad f < f_c. \tag{10–58}$$

Substitution of Eq. (10–58) in Eq. (10–54) gives the wave impedance of TE modes for $f < f_c$:

$$Z_{TE} = j\frac{\omega\mu}{h\sqrt{1 - (f/f_c)^2}}, \qquad f < f_c, \tag{10–59}$$

which is purely reactive, indicating again that there is no power flow for evanescent waves at $f < f_c$.

The wave impedance Z is reactive (no power flow with evanescent waves)

The three types of modes

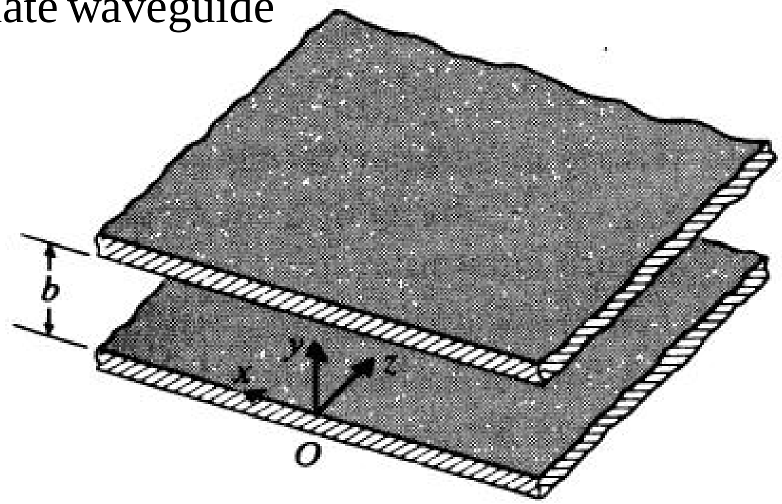
TABLE 10–1
Wave Impedances and Guide Wavelengths for $f > f_c$

Mode	Wave Impedance, Z	Guide Wavelength, λ_g
TEM	$\eta = \sqrt{\dfrac{\mu}{\epsilon}}$	$\lambda = \dfrac{1}{f\sqrt{\mu\epsilon}}$
TM	$\eta\sqrt{1 - \left(\dfrac{f_c}{f}\right)^2}$	$\dfrac{\lambda}{\sqrt{1-(f_c/f)^2}}$
TE	$\dfrac{\eta}{\sqrt{1-(f_c/f)^2}}$	$\dfrac{\lambda}{\sqrt{1-(f_c/f)^2}}$

Note:
1) Wavelength, phase velocity, group velocity are the same for TE and TM waves.
2) The wave impedance Z of TE waves is larger than the dielectric impedance, while the wave impedance Z of TM waves is smaller than the dielectric impedance.

Parallel-Plate Waveguide

A infinite parallel-plate waveguide



Parallel-plate waveguide of two perfectly conducting plates separated by a dielectric medium with constitutive parameters ε and μ.

Note:

Perfect conductors: the electric conductivity σ of the parallel plates is infinite.

Perfect dielectrics: the electric conductivity σ of the filling medium is zero.

TM Waves in a Parallel-Plate Waveguide

Since E, H are expressed in terms of Ez, for the first, we need to obtain Ez:

$$\frac{d^2E_z^0(y)}{dy^2} + h^2E_z^0(y) = 0. \tag{10–62}$$

The solution of Eq. (10–62) must satisfy the boundary conditions

$$E_z^0(y) = 0 \qquad \text{at } y = 0 \quad \text{and} \quad y = b.$$

From Section 4–5 we conclude that $E_z^0(y)$ must be of the following form ($h = n\pi/b$):

$$E_z^0(y) = A_n \sin\left(\frac{n\pi y}{b}\right), \tag{10–63}$$

The solution is given by

$$H_x^0(y) = \frac{j\omega\epsilon}{h} A_n \cos\left(\frac{n\pi y}{b}\right)$$

$$E_y^0(y) = -\frac{\gamma}{h} A_n \cos\left(\frac{n\pi y}{b}\right)$$

The wave is uniform (i.e., it does not vary) in the x-direction; The wave is a standing wave in the y-direction;The wave is propagating (travelling) in the z-direction.

The cutoff frequency is determined by the wavenumber of the standing wave in the y-direction.

The γ in Eq. (10–65) is the propagation constant that can be determined from Eq. (10–33):

$$\gamma = \sqrt{\left(\frac{n\pi}{b}\right)^2 - \omega^2\mu\epsilon}. \qquad (10\text{–}66)$$

The cutoff frequency is the frequency that makes $\gamma = 0$. We have

$$\boxed{f_c = \frac{n}{2b\sqrt{\mu\epsilon}} \qquad \text{(Hz)},} \qquad (10\text{–}67)$$

Waves with f>fc propagate with a phase constant βgiven in Eq. (10-38), while waves with f<fc are evanescent.

TM for n=0 exists (In fact, TM for n=0 is a TEM wave because its Ez also vanishes).
Obviously, for n=0, the cutoff frequency fc=0.
The mode having the lowest cutoff frequency is called the “dominant mode”.
Thus, for a parallel-plate waveguide, the dominant mode is the TEM mode.

TE Waves in a Parallel-Plate Waveguide

Since E, H are expressed in terms of Hz, for the first, we need to obtain Hz:

10–3.2 TE WAVES BETWEEN PARALLEL PLATES

For transverse electric waves, $E_z = 0$, we solve the following equation for $H_z^0(y)$, which is a simplified version of Eq. (10–48) with no x-dependence:

$$\frac{d^2 H_z^0(y)}{dy^2} + h^2 H_z^0(y) = 0. \tag{10–82}$$

We note that $H_z(y, z) = H_z^0(y)e^{-\gamma z}$. The boundary conditions to be satisfied by $H_z^0(y)$ are obtained from Eq. (10–51). Since E_x must vanish at the surfaces of the conducting plates, we require

$$\frac{dH_z^0(y)}{dy} = 0 \qquad \text{at } y = 0 \quad \text{and} \quad y = b.$$

The wave is uniform (i.e., it does not vary) in the x-direction; The wave is a standing wave in the y-direction;

The wave is propagating (travelling) in the z-direction.

Therefore the proper solution of Eq. (10–82) is of the form

$$H_z^0(y) = B_n \cos\left(\frac{n\pi y}{b}\right), \tag{10–83}$$

where the amplitude B_n depends on the strength of excitation of the particular TE wave. We obtain the only other nonzero field components from Eqs. (10–50) and (10–51), keeping in mind that $\partial H_z/\partial x = 0$:

$$H_y^0(y) = \frac{\gamma}{h} B_n \sin\left(\frac{n\pi y}{b}\right), \tag{10–84}$$

$$E_x^0(y) = \frac{j\omega\mu}{h} B_n \sin\left(\frac{n\pi y}{b}\right). \tag{10–85}$$

Note:

1) The cutoff frequency for the TEn mode in a parallel-plate waveguide is exactly the same as that for the TMn mode given in Eq. (10-67).
2) TEn for n=0 does not exist, because Hy=0, Ex=0.
3) TMn for n=0 exists (In fact, TM for n=0 is a TEM wave because its Ez=0).

TM waves in rectangular waveguides

$$E_z^0(x, y) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right) \qquad \text{(V/m)}$$

$$E_x^0(x, y) = -\frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)E_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

$$E_y^0(x, y) = -\frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)E_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right),$$

$$H_x^0(x, y) = \frac{j\omega\epsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right),$$

$$H_y^0(x, y) = -\frac{j\omega\epsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right),$$

$$\gamma = j\beta = j\sqrt{\omega^2\mu\epsilon - \left(\frac{m\pi}{a}\right)^2 - \left(\frac{n\pi}{b}\right)^2}. \qquad h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2.$$

Note: When the dimension a approaches an infinite value, Ex and Hy are zero, and Ey and Hx are reduced to the forms in (10-64), (10-65) of TM modes in parallel-plates waveguides.

For the TM_{mn} mode the cutoff frequency is

$$\boxed{(f_c)_{mn} = \frac{1}{2\sqrt{\mu\epsilon}} \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2} \qquad \text{(Hz)},}$$

$$\boxed{(\lambda_c)_{mn} = \frac{2}{\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}} \qquad \text{(m)},}$$

where λ_c is the ***cutoff wavelength.***

For TM modes in rectangular waveguides, neither m nor n can be zero.
Hence the TM11 mode has the lowest cutoff frequency of all TM modes in a rectangular waveguide.

TE waves in rectangular waveguides

Since E, H are expressed in terms of Hz, for the first, we need to obtain Hz:

$$H_z^0(x, y) = H_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right) \qquad \text{(A/m)}.$$

$$E_x^0(x, y) = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \qquad (10$$

$$E_y^0(x, y) = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), \qquad (10$$

$$H_x^0(x, y) = \frac{\gamma}{h^2}\left(\frac{m\pi}{a}\right)H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right), \qquad (10$$

$$H_y^0(x, y) = \frac{\gamma}{h^2}\left(\frac{n\pi}{b}\right)H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right), \qquad (10$$

where γ has the same expression as that given in Eq. (10–138) for TM modes.

Note: When the dimension a approaches an infinite value, Ey and Hx are zero, and Ex and Hy are reduced to the forms in (10-84), (10-85) of TE modes in parallel-plates waveguides

$m = 1$ and $n = 0$:

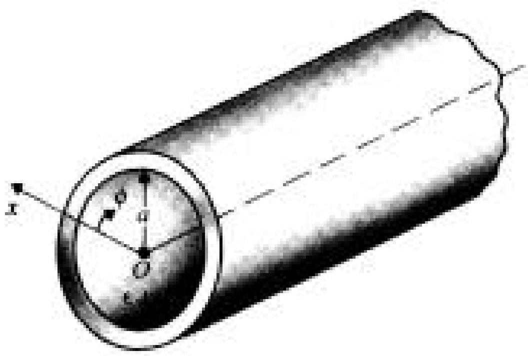
$$\boxed{(f_c)_{\mathrm{TE}_{10}} = \frac{1}{2a\sqrt{\mu\epsilon}} = \frac{u}{2a} \qquad \text{(Hz)}.} \tag{10-163}$$

The corresponding cutoff wavelength is

$$\boxed{(\lambda_c)_{\mathrm{TE}_{10}} = 2a \qquad \text{(m)}.} \tag{10-164}$$

Hence ***the*** **TE$_{10}$** ***mode is the dominant mode of a rectangular waveguide with a > b.***

Circular Waveguides



Circular Waveguides: Metal pipes having a uniform circular cross section and being filled with a dielectric medium.

A straight waveguide with a uniform circular cross section (its axis in the z-direction)

The TEM waves cannot exist in such a waveguide without an inner conductor.
The propagating waves can be classified into two groups: transverse magnetic (TM) and transverse electric (TE) waves.

For the TM waves, we need to solve the homogeneous Helmholtz's equation of Ez.

Bessel's differential equation and Bessel functions

In cylindrical coordinates the expansion of Eq. (10-196) gives (see Eq. 4-8)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E_z^0}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 E_z^0}{\partial \phi^2}+h^2E_z^0=0. \qquad (10\text{-}197)$$

To solve Eq. (10-197), we apply the method of separation of variables by assuming a product solution.

$$E_z^0(r,\phi)=R(r)\Phi(\phi), \qquad (10\text{-}198)$$

where *R(r)* and $\Phi(\phi)$ are functions only of r and ϕ, respectively. Substituting solution (10-198) in Eq. (10-197) and dividing by the product $R(r)\Phi(\phi)$, we obtain

$$\frac{r}{R(r)}\frac{d}{dr}\left[r\frac{dR(r)}{dr}\right]+h^2r^2=-\frac{1}{\Phi(\phi)}\frac{d^2\Phi(\phi)}{d\phi^2}. \qquad (10\text{–}199)$$

Now the left side of Eq. (10-199) is a function of r only, and the right side is a function of ϕ only. For Eq. (10-199) to hold for all values of r and ϕ, both sides must be equal to the same constant. Let this constant (separation constant) be n^2.

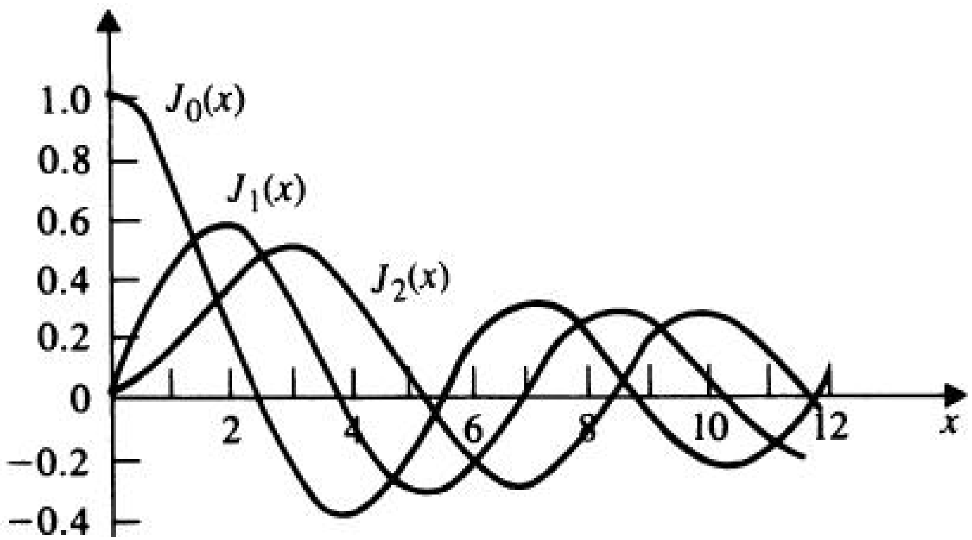
We can separate Eq. (10-199) into two ordinary differential equations:

$$\frac{d^2\Phi(\phi)}{d\phi^2} + n^2\Phi(\phi) = 0 \qquad (10\text{-}200)$$

$$\boxed{\frac{d^2R(r)}{dr^2} + \frac{1}{r}\frac{dR(r)}{dr} + \left(h^2 - \frac{n^2}{r^2}\right)R(r) = 0.} \qquad (10\text{–}201)$$

Equation (10–201) is known as ***Bessel's differential equation.***

The Bessel functions of the first kind



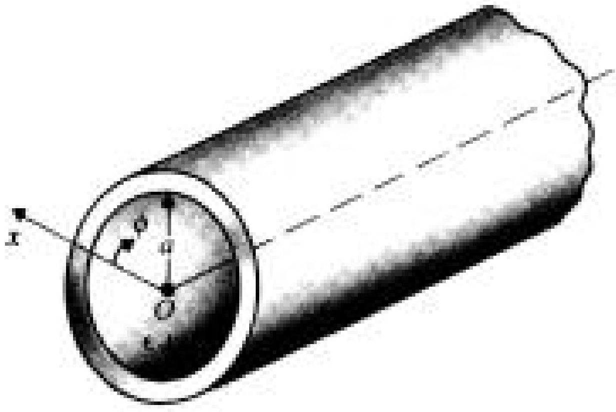
The Bessel functions Jn(x) of the first kind have three properties:

①Jn(0)=0 for all n except when n=0. For n=0, J0(0)=1.

②Jn(x) are alternating functions of decreasing amplitudes.

③As x becomes large, the Bessel functions Jn(x) approach the sinusoidal forms.

TM waves in circular waveguides



10–5.2 TM WAVES IN CIRCULAR WAVEGUIDES

Figure 10–19 shows a circular waveguide of radius a. It consists of a metal pipe centered around the z-axis. The enclosed dielectric medium is assumed to have constitutive parameters ϵ and μ. For TM waves, $H_z = 0$. We write

$$E_z(r, \phi, z) = E_z^0(r, \phi)e^{-\gamma z}, \tag{10–216}$$

where $E_z^0(r, \phi)$ satisfies Eq. (10–196). The solution is written in the form of Eq. (10–198), in which

$$R(r) = C_n J_n(hr), \tag{10–217}$$

The solutions for TM waves in the circular waveguide

$$E_z^0 = C_n J_n(hr)\cos n\phi. \qquad \text{(TM modes)}$$

$$E_r^0 = -\frac{j\beta}{h} C_n J_n'(hr)\cos n\phi,$$

$$E_\phi^0 = \frac{j\beta n}{h^2 r} C_n J_n(hr)\sin n\phi,$$

$$H_r^0 = -\frac{j\omega\epsilon n}{h^2 r} C_n J_n(hr)\sin n\phi,$$

$$H_\phi^0 = -\frac{j\omega\epsilon}{h} C_n J_n'(hr)\cos n\phi,$$

$$H_z^0 = 0,$$

eigenvalue for the TM_{01} mode that corresponds to the first zero ($x_{01} = 2.405$) of $J_0(x)$ is

$$(h)_{TM_{01}} = \frac{2.405}{a}, \tag{10-227}$$

which yields the lowest cutoff frequency for a TM mode:

$$(f_c)_{TM_{01}} = \frac{(h)_{TM_{01}}}{2\pi\sqrt{\mu\epsilon}} = \frac{0.383}{a\sqrt{\mu\epsilon}}. \tag{10-228}$$

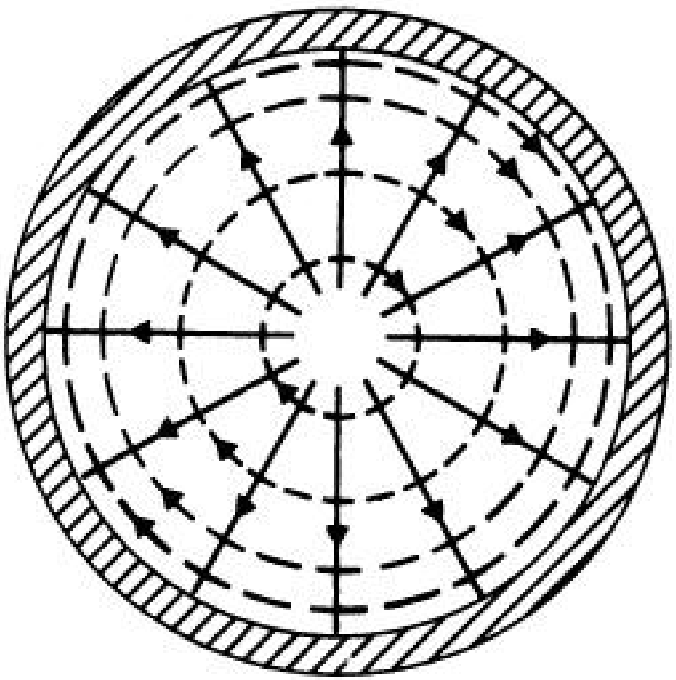


Fig. 10-20
Field lines for TM01 mode in a transverse plane of circular waveguide

TM01 mode: Ez, Er and Hφ are the only nonzero field components.

The solutions for TE waves in the circular waveguide

10–5.3 TE WAVES IN CIRCULAR WAVEGUIDES

For TE modes, $E_z = 0$, and

$$H_z(r, \phi, z) = H_z^0(r, \phi)e^{-\gamma z},$$

where H_z^0 satisfies the homogeneous Helmholtz's equation

$$\nabla_{r\phi}^2 H_z^0 + h^2 H_z^0 = 0.$$

Analogously to the TM case, we write the solution as

$$\boxed{H_z^0 = C_n' J_n(hr) \cos n\phi. \qquad \text{(TE modes)}}$$

By using Eqs. (10-49) (10-50) (10-51) (10-52), we can
obtain all the field components of TE modes in the circular waveguide:

The solutions for TE waves in the circular waveguide

$$H_r^0 = -\frac{j\beta}{h} C_n' J_n'(hr) \cos n\phi,$$

$$H_\phi^0 = \frac{j\beta n}{h^2 r} C_n' J_n(hr) \sin n\phi,$$

$$E_r^0 = \frac{j\omega\mu n}{h^2 r} C_n' J_n(hr) \sin n\phi,$$

$$E_\phi^0 = \frac{j\omega\mu}{h} C_n' J_n'(hr) \cos n\phi,$$

$$E_z^0 = 0.$$

The first several zeros of $J_n'(x)$ are listed in Table 10–3, from which we see that the smallest x'_{np} is $x'_{11} = 1.841$. This corresponds to the *smallest eigenvalue*

$$(h)_{\mathrm{TE}_{11}} = \frac{1.841}{a}, \tag{10–238}$$

and the *lowest cutoff frequency*

$$\boxed{(f_c)_{\mathrm{TE}_{11}} = \frac{h_{\mathrm{TE}_{11}}}{2\pi\sqrt{\mu\epsilon}} = \frac{0.293}{a\sqrt{\mu\epsilon}} \qquad \text{(Hz)},} \tag{10–239}$$

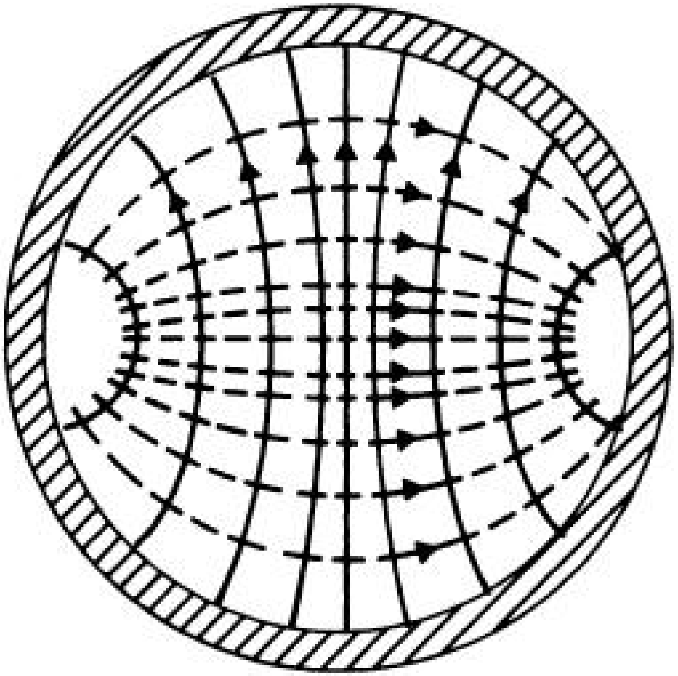
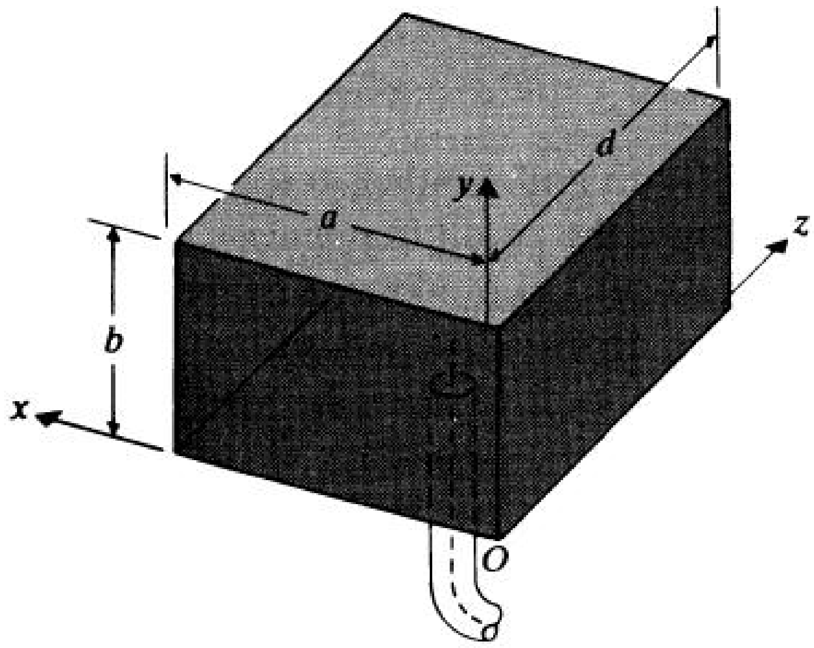


Fig. 10-21
Field lines for TE11 mode in a transverse plane of a circular waveguide

——— Electric field lines

— — — Magnetic field lines

Cavity Resonators



An enclosed conducting box, which is essentially a segment of a waveguide with closed end faces, is called a cavity resonator.

TMmnp modes in a rectangular cavity resonator

The phasor of the field components:

$$E_z(x, y, z) = E_0 \sin\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right), \tag{10–295}$$

$$E_x(x, y, z) = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)E_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right), \tag{10–296}$$

$$E_y(x, y, z) = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)E_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right), \tag{10–297}$$

$$H_x(x, y, z) = \frac{j\omega\epsilon}{h^2}\left(\frac{n\pi}{b}\right)E_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right), \tag{10–298}$$

$$H_y(x, y, z) = -\frac{j\omega\epsilon}{h^2}\left(\frac{m\pi}{a}\right)E_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right), \tag{10–299}$$

where

$$h^2 = \left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2. \tag{10–300}$$

TEmnp modes in a rectangular cavity resonator

The phasor of the field components:

$$H_z(x, y, z) = H_0 \cos\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right), \tag{10–302}$$

$$E_x(x, y, z) = \frac{j\omega\mu}{h^2}\left(\frac{n\pi}{b}\right)H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right), \tag{10–303}$$

$$E_y(x, y, z) = -\frac{j\omega\mu}{h^2}\left(\frac{m\pi}{a}\right)H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\sin\left(\frac{p\pi}{d}z\right), \tag{10–304}$$

$$H_x(x, y, z) = -\frac{1}{h^2}\left(\frac{m\pi}{a}\right)\left(\frac{p\pi}{d}\right)H_0 \sin\left(\frac{m\pi}{a}x\right)\cos\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right), \tag{10–305}$$

$$H_y(x, y, z) = -\frac{1}{h^2}\left(\frac{n\pi}{b}\right)\left(\frac{p\pi}{d}\right)H_0 \cos\left(\frac{m\pi}{a}x\right)\sin\left(\frac{n\pi}{b}y\right)\cos\left(\frac{p\pi}{d}z\right), \tag{10–306}$$

From Eq. (10–138) we obtain the following expression for the resonant frequency:

$$\omega_{mnp} = \frac{1}{\sqrt{\mu\epsilon}}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{p\pi}{d}\right)^2}$$

or

$$\boxed{f_{mnp} = \frac{u}{2}\sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2 + \left(\frac{p}{d}\right)^2} \quad \text{(Hz).}} \tag{10–301}$$