BEC301 SIGNALS AND SYSTEMS

UNIT-I

CLASSIFICATION OF SIGNALS AND SYSTEMS

Recommended Reading Material

1. Allan V.Oppenheim, S.Wilsky and S.H.Nawab, “Signals and Systems”, Pearson, 2007.
2. 2. B. P. Lathi, “Principles of Linear Systems and Signals”, Second Edition, Oxford, 2009.

CLASSIFICATION OF SIGNALS AND SYSTEMS

- What is a Signal?

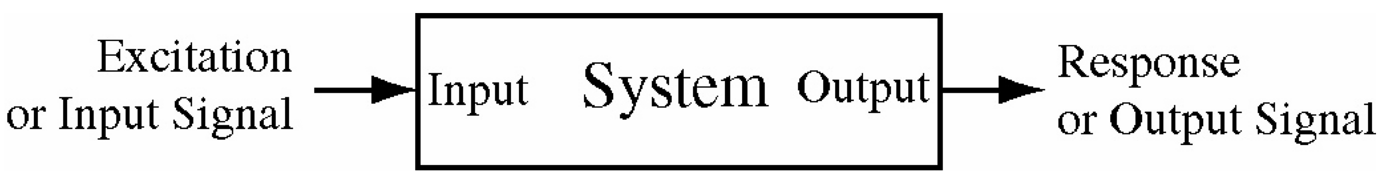
  A signal is defined as a time varying physical phenomenon which conveys information

Examples :Electrical signals, Acoustic signals, Voice signals, Video signals, EEG, ECG etc.

- What is a System?

  System is a device or combination of devices, which can operate on signals and produces corresponding response.

- Input to a system is called as excitation and output from it is called as response.



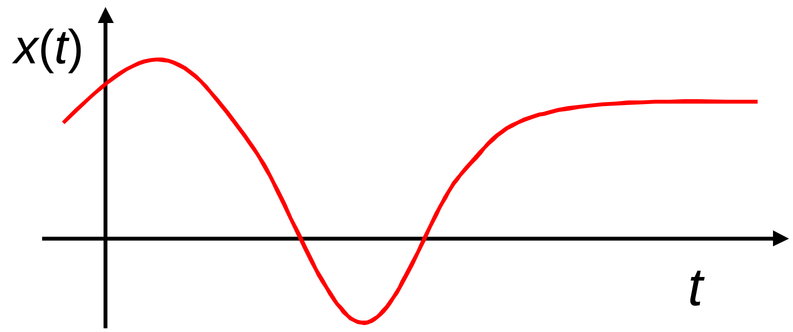
Continuous & Discrete-Time Signals

Continuous-Time Signals

Most signals in the real world are continuous time, as the scale is infinitesimally fine.

Eg voltage, velocity,

Denote by $x(t)$, where the time interval may be bounded (finite) or infinite

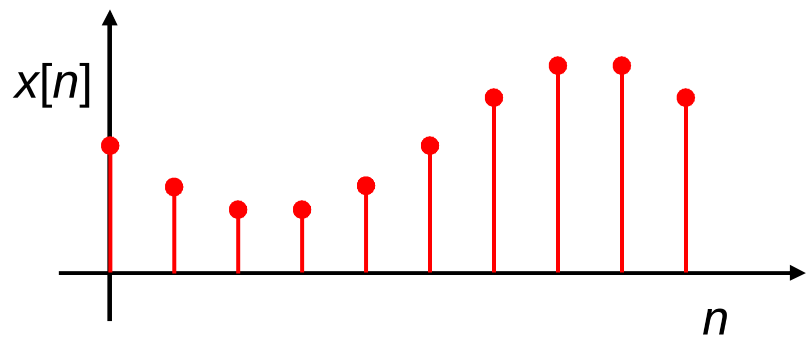


Discrete-Time Signals

Some real world and many digital signals are discrete time, as they are sampled

E.g. pixels, daily stock price (anything that a digital computer processes)

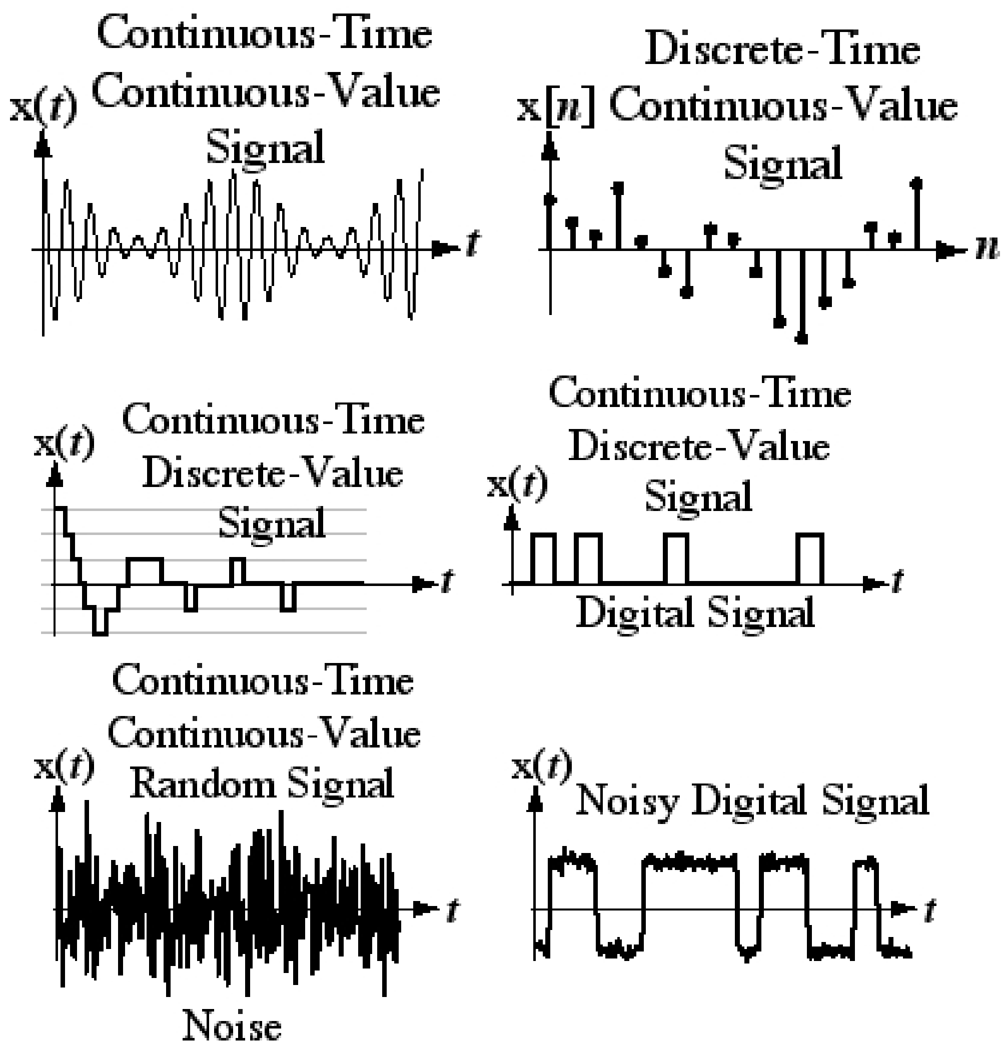
Denote by $x[n]$, where n is an integer value that varies discretely



Sampled continuous signal

$x[n] = x(nk)$ – k is sample time

Signal Types



Signal classification

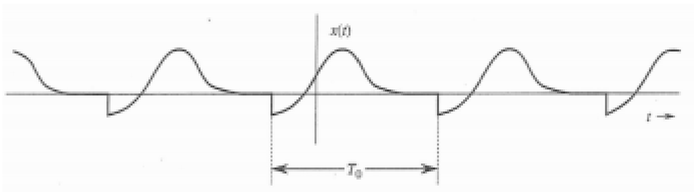
Signals may be classified into:

1. Periodic and aperiodic signals
2. Energy and power signals
3. Deterministic and probabilistic signals
4. Causal and non-causal
5. Even and Odd signals

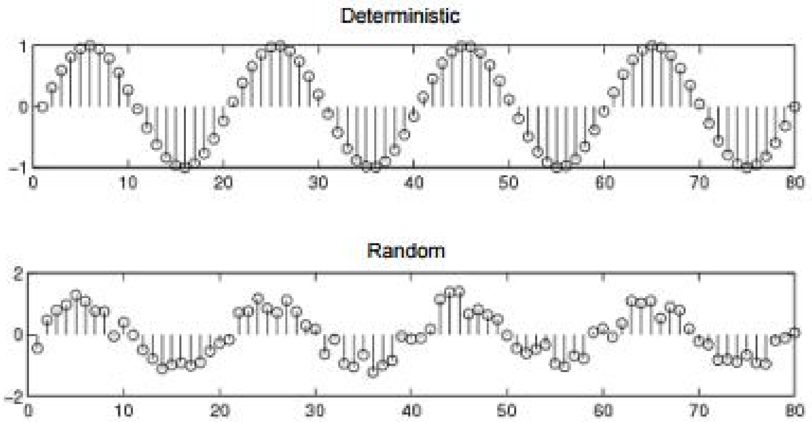
Signal Properties

Periodic signals: a signal is periodic if it repeats itself after a fixed period T, i.e. $x(t) = x(t+T_0)$ for all t. A $\sin(t)$ signal is periodic.

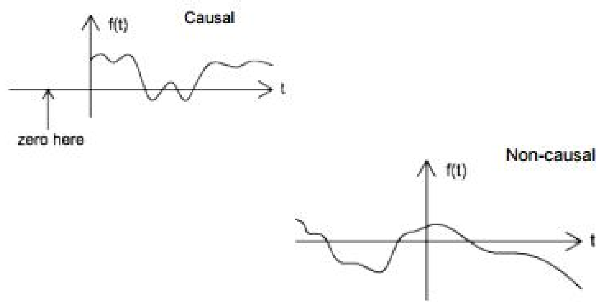
The smallest value of To that satisfies the periodicity condition of this equation is the fundamental period of x(t).



Deterministic and Random Signals:



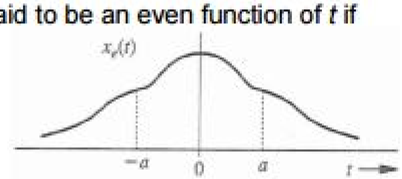
Causal and Non-Causal Signals:



Even and odd signals: a signal is even if x(-t) = x(t) (i.e. it can be reflected in the axis at zero). A signal is odd if x(-t) = -x(t). Examples are cos(t) and sin(t) signals, respectively

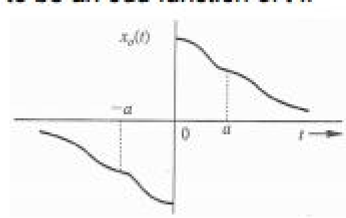
- A real function $x_e(t)$ is said to be an even function of *t* if

$$x_e(t) = x_e(-t)$$



- A real function $x_o(t)$ is said to be an odd function of *t* if

$$x_o(t) = -x_o(-t)$$



Decomposition in even and odd components

- Any signal can be written as a combination of an even and an odd signals – Even and odd components

$$f(t) = \frac{1}{2}\left(f(t) + f(-t)\right) + \frac{1}{2}\left(f(t) - f(-t)\right)$$

$$f_e(t) = \frac{1}{2}\left(f(t) + f(-t)\right) \quad \text{even component}$$

$$f_o(t) = \frac{1}{2}\left(f(t) - f(-t)\right) \quad \text{odd component}$$

$$f(t) = f_e(t) + f_o(t)$$

Energy and Power signal

A signal with finite energy is an energy signal

$$E_f = \int_{-\infty}^{+\infty} |f(t)|^2 \, dt$$

A signal with finite and different from zero power is a power signal

Power – The power is the time average (mean) of the squared signal amplitude, that is the mean-squared value of f(t) .

$$P_f = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{+T/2} |f(t)|^2 \, dt$$

There exists signals for which neither the energy nor the power are finite .

Elementary Signals

Unit Step Signal:

Useful for representing causal signals

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

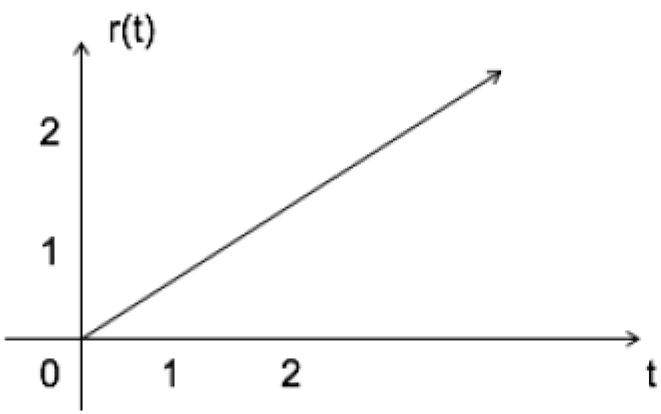
The discrete-time unit step signal u[n] is defined as

$$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases}$$

Ramp Signal

Ramp signal is denoted by r(t), and it is defined as

$$r(t) = \begin{cases} 0 \text{ if } t < 0 \\ \frac{t}{t_0} \text{ if } 0 \le t \le t_0 \\ 1 \text{ if } t > t_0 \end{cases}$$

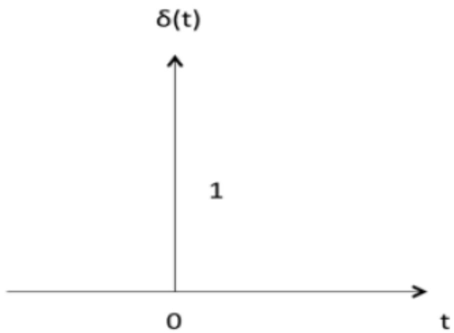


$$u(t) = \frac{dr(t)}{dt}$$

Area under unit ramp is unity.

Unit Impulse Function

Impulse function is denoted by δ(t).

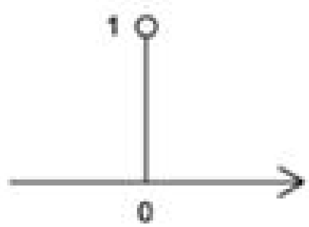


$$\int_{-\infty}^{\infty} \delta(t)dt = u(t)$$

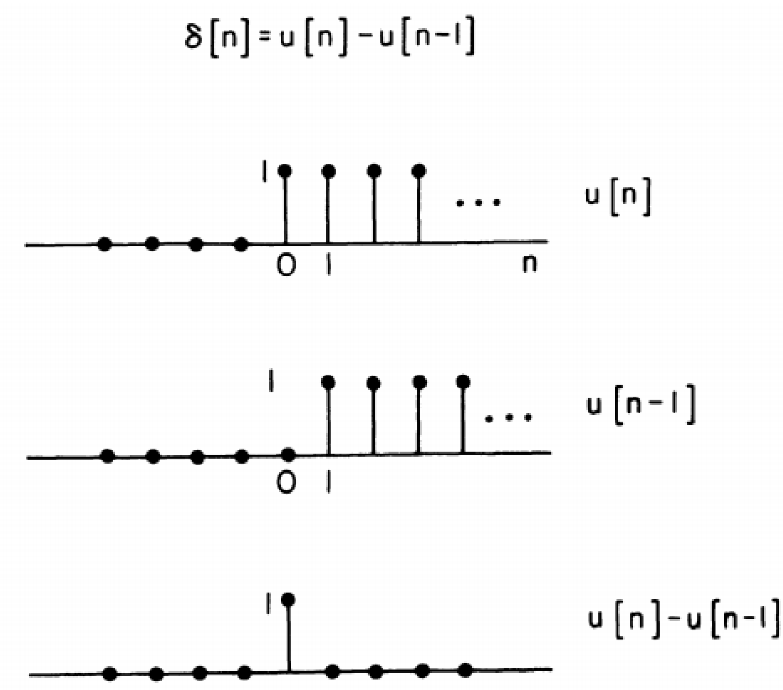
$$\delta(t) = \frac{du(t)}{dt}$$

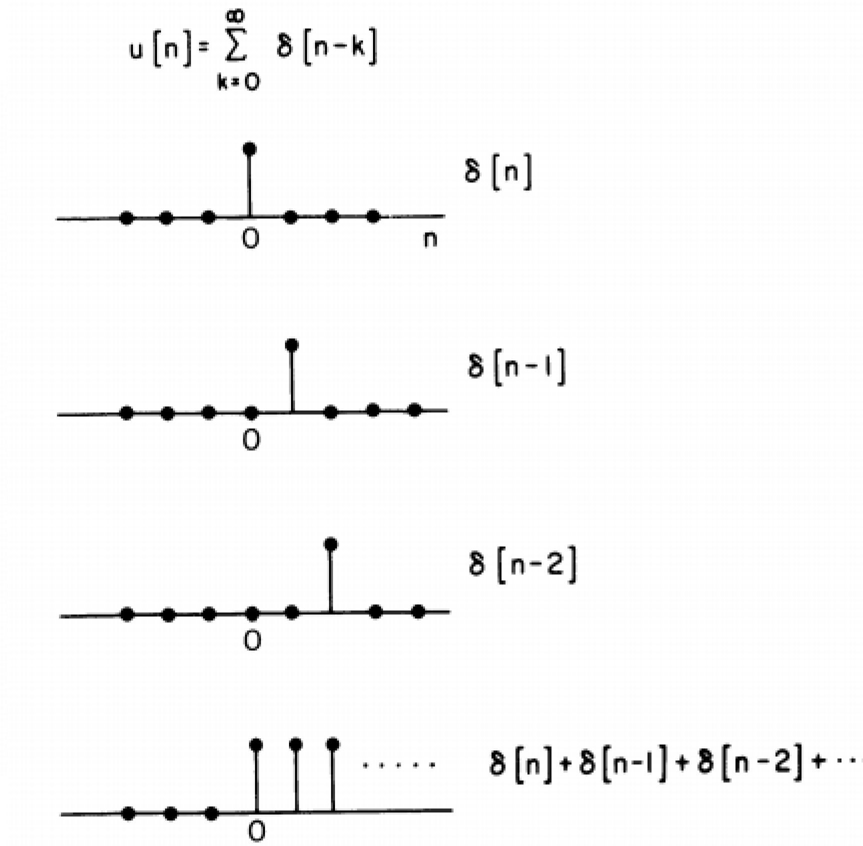
Discrete time impulse function

$$\delta[n] = \begin{cases} 1 \text{ if } n = 0 \\ 0 \text{ otherwise} \end{cases}$$



Relation Between the Elementary Signals

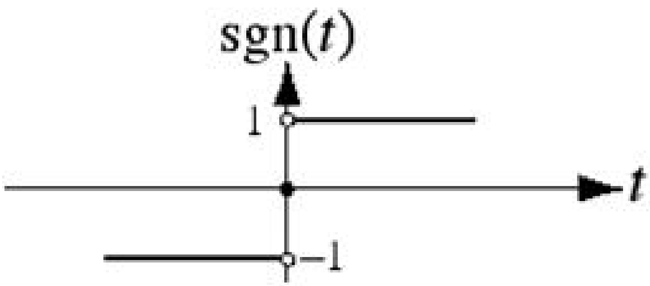




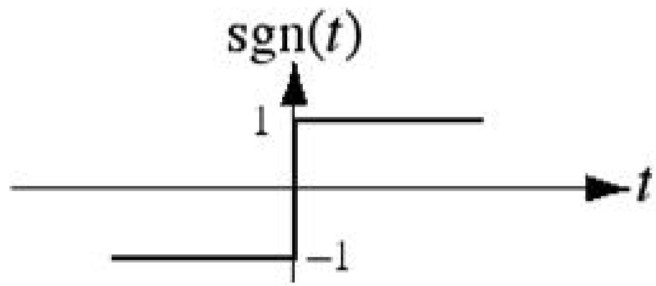
Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & , \ t > 0 \\ 0 & , \ t = 0 \\ -1 & , \ t < 0 \end{cases}$$

Precise Graph



Commonly-Used Graph



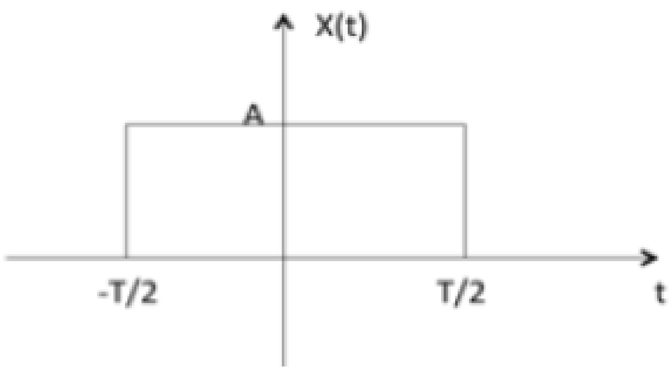
The signum function, in a sense, returns an indication of the sign of its argument.

Elementary Signals

Rectangular Signal

Let it be denoted as x(t)

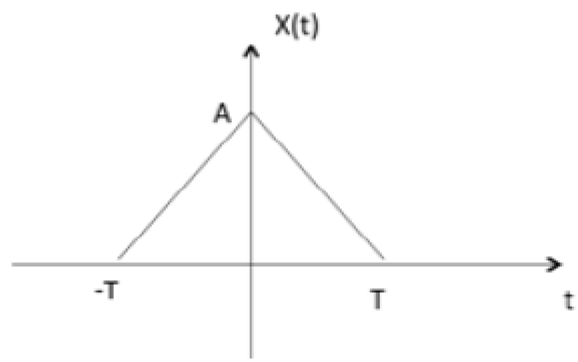
$$x(t) = A\, rect\left[\frac{r}{T}\right]$$



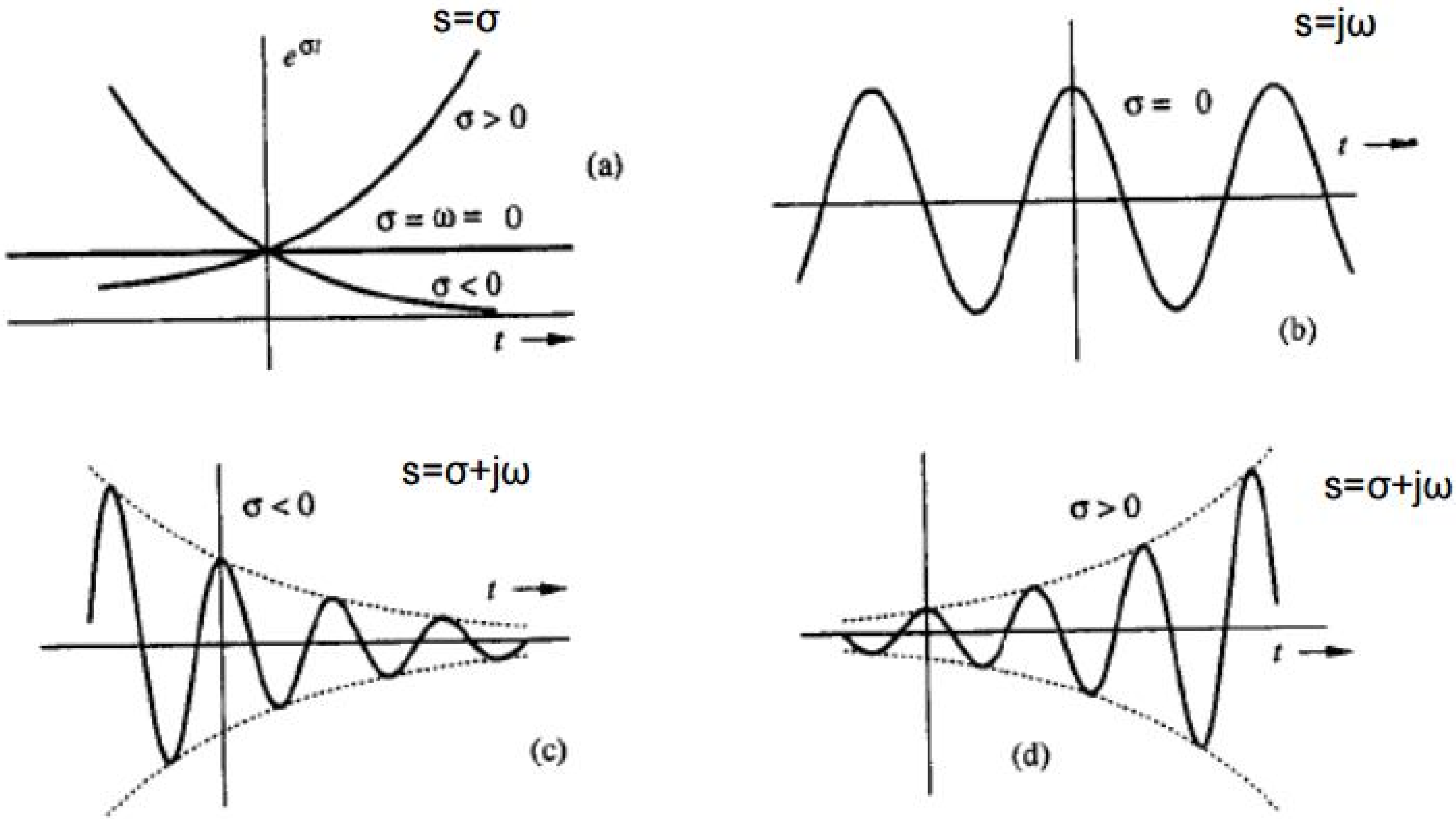
Triangular Signal

Let it be denoted as x(t)

$$x(t) = A\left[1 - \frac{|t|}{T}\right]$$



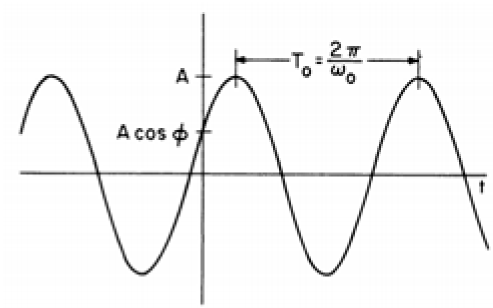
Exponential and Sinusoidal Signals



Sinusoidal Signals

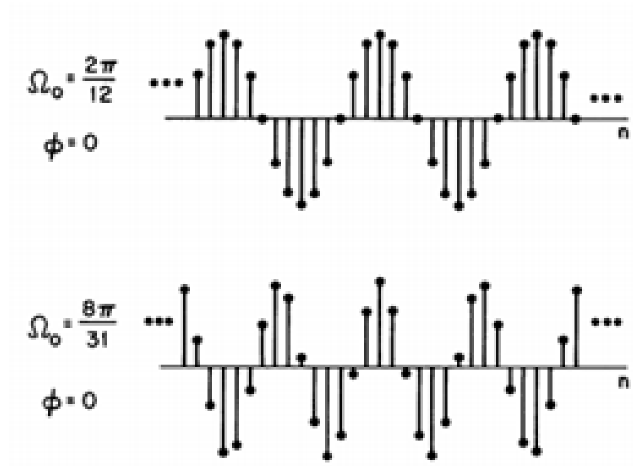
Continuous Time sinusoidal signal

$$x(t) = A\cos(\omega_0 t + \phi)$$



Discrete Time sinusoidal signal

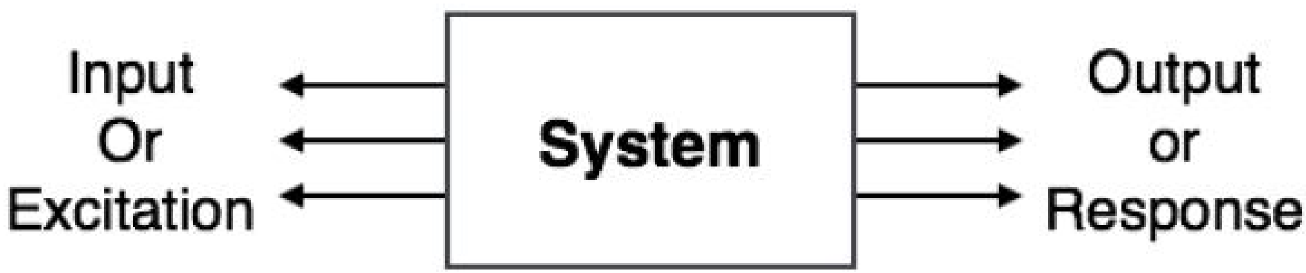
$$x[n] = A\cos(\Omega_0 n + \phi)$$



Systems

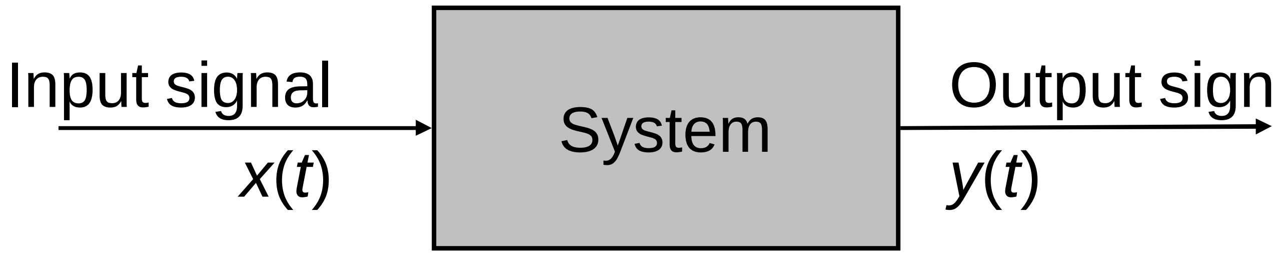
A system is characterized by

- inputs
- outputs
- rules of operation (mathematical model of the system)



How is a System Represented?

A system takes a signal as an input and transforms it into another signal



In a very broad sense, a system can be represented as the ratio of the output signal over the input signal

That way, when we “multiply” the system by the input signal, we get the output signal.

$$y(t)= F\ (x(t))$$

Classification of Systems

Systems are classified into the following categories:

- Linear and Non-linear Systems
- Time Variant and Time Invariant Systems
- Static and Dynamic Systems
- Causal and Non-causal Systems
- Invertible and Non-Invertible Systems
- Stable and Unstable Systems

Linear and Non-Linear System

A system is said to be linear when it satisfies superposition and homogenate principles.

Consider two systems with inputs as x1(t), x2(t), and outputs as y1(t), y2(t) respectively.

Then, according to the superposition and homogenate principles,

$$T\,[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, T[x_1(t)] + a_2\, T[x_2(t)]$$

$$\therefore\ T\,[a_1\, x_1(t) + a_2\, x_2(t)] = a_1\, y_1(t) + a_2\, y_2(t)$$

Thus response of overall system is equal to response of the individual system.

Time/Shift Invariant

Time-invariance: A system is time invariant if the system's output is the same, given the same input signal, regardless of time.

$$g_i(x) = H[f_i(x)] \text{ implies that } g_i(x+x_0) = H[f_i(x+x_0)]$$

for all $f_i(x) \in \{f(x)\}$ and for all x_0.

Offsetting the independent variable of the input by x0 causes the same offset in the independent variable of the output. Hence the input-output relation remains the same.

Static and Dynamic Systems

Static system is memory-less whereas dynamic system is a memory system.

Example 1: $y(t) = 2\,x(t)$

For present value t=0, the system output is $y(0) = 2x(0)$.

Here, the output is only dependent upon present input.

Hence the system is memory less or static.

Example 2: $y(t) = 2\,x(t) + 3\,x(t-3)$

For present value t=0, the system output is $y(0) = 2x(0) + 3x(-3)$.

Here $x(-3)$ is past value for the present input for which the system requires memory to get this output.

Hence, the system is a dynamic system.

Causal and Non-Causal Systems

A system is said to be causal if its output depends upon present and past inputs, and does not depend upon future input.

For non causal system, the output depends upon future inputs also.

$$\boxed{f(x) = 0 \text{ for } x < x_0 \text{ implies that } g(x) = H[f(x)] \;\; = 0 \text{ for } x < x_0.}$$

Example : y(n) = 2 x(t) + 3 x(t-3)

For present value t=1,

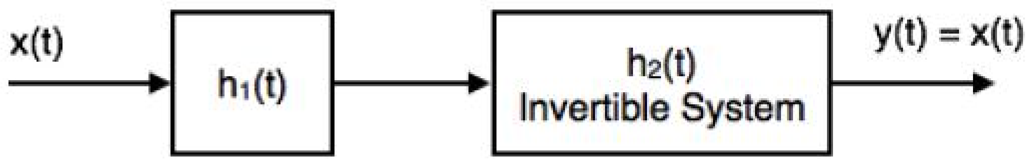
the system output is y(1) = 2x(1) + 3x(-2).

Here, the system output only depends upon present and past inputs.

Hence, the system is causal.

Invertible and Non-Invertible systems

A system is said to invertible if the input of the system appears at the output.



$Y(S) = X(S)\ H1(S)\ H2(S) = X(S)\ H1(S) \cdot 1/(H1(S))$

Since $H2(S) = 1/(\ H1(S)\ )$

$\therefore\ Y(S) = X(S)$

$\rightarrow\ y(t) = x(t)$

Hence, the system is invertible.

If $y(t) \neq x(t)$, then the system is said to be non-invertible

Stable and Unstable Systems

The system is said to be stable only when the output is bounded for bounded input. For a bounded input, if the output is unbounded in the system then it is said to be unstable.

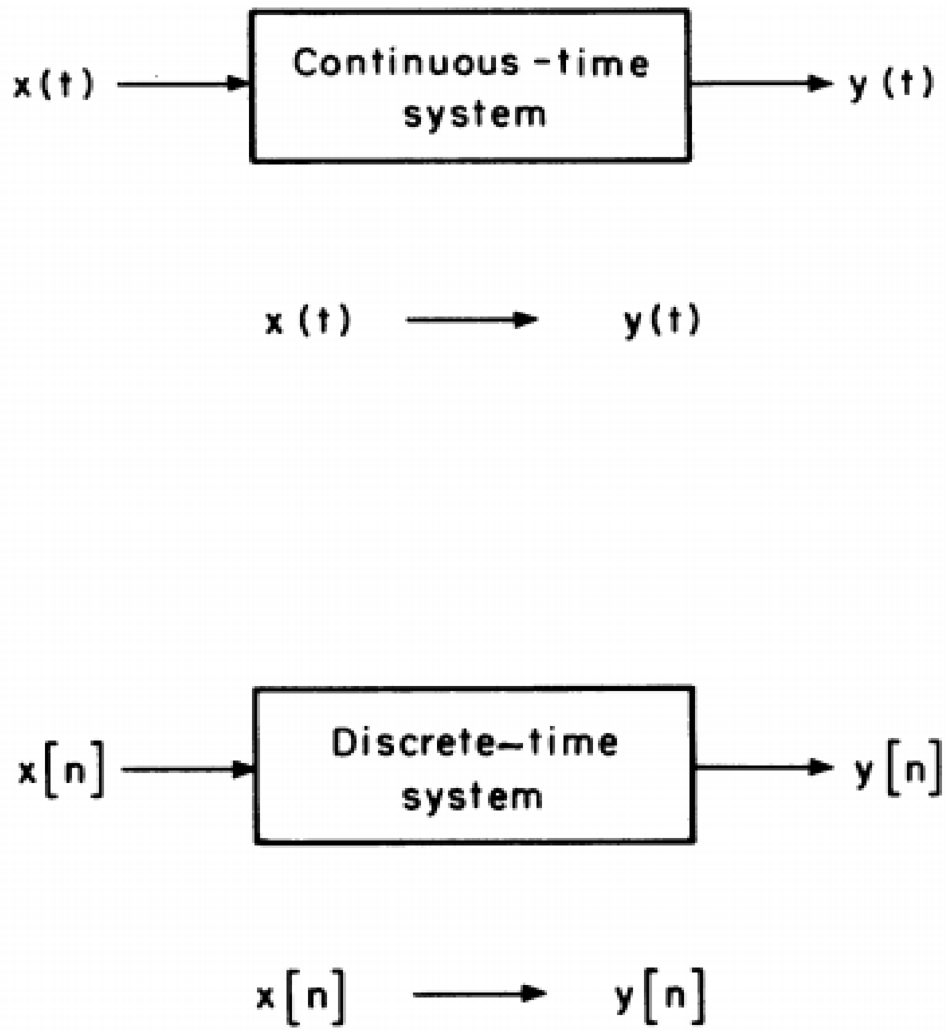
$$\boxed{|f(x)| < K \text{ implies that } |g(x)| < cK}$$

Example : $y(t) = x^2(t)$

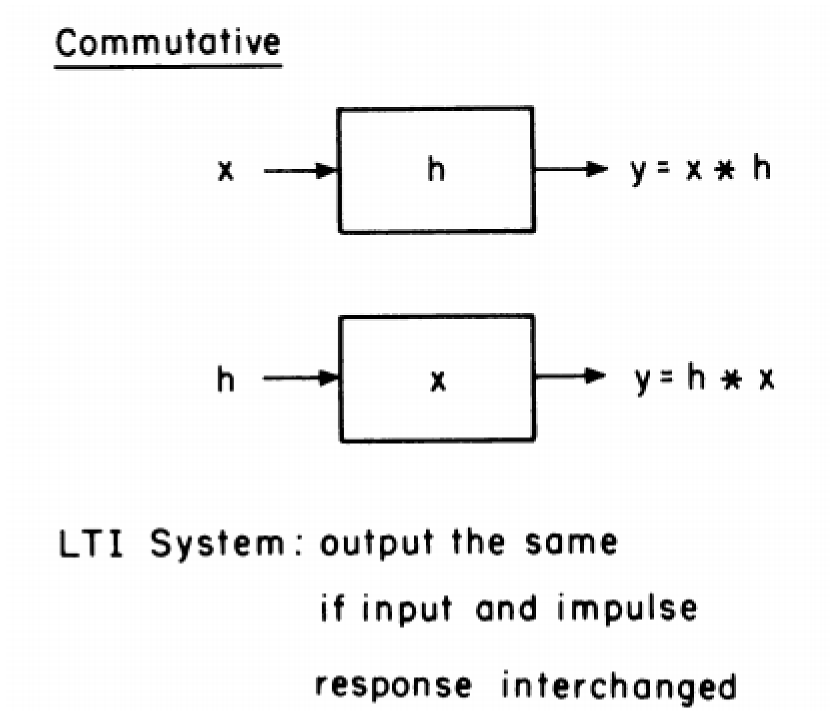
Let the input is $u(t)$ (unit step bounded input) then the output $y(t) = u^2(t) = u(t)$ = bounded output.

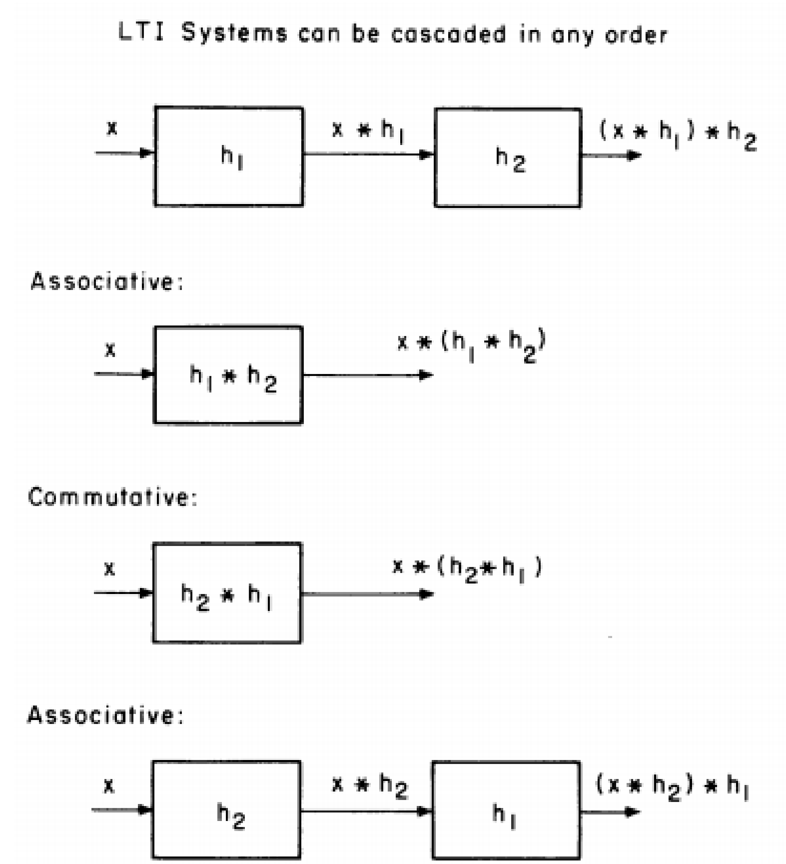
Hence, the system is stable.

Types of Systems

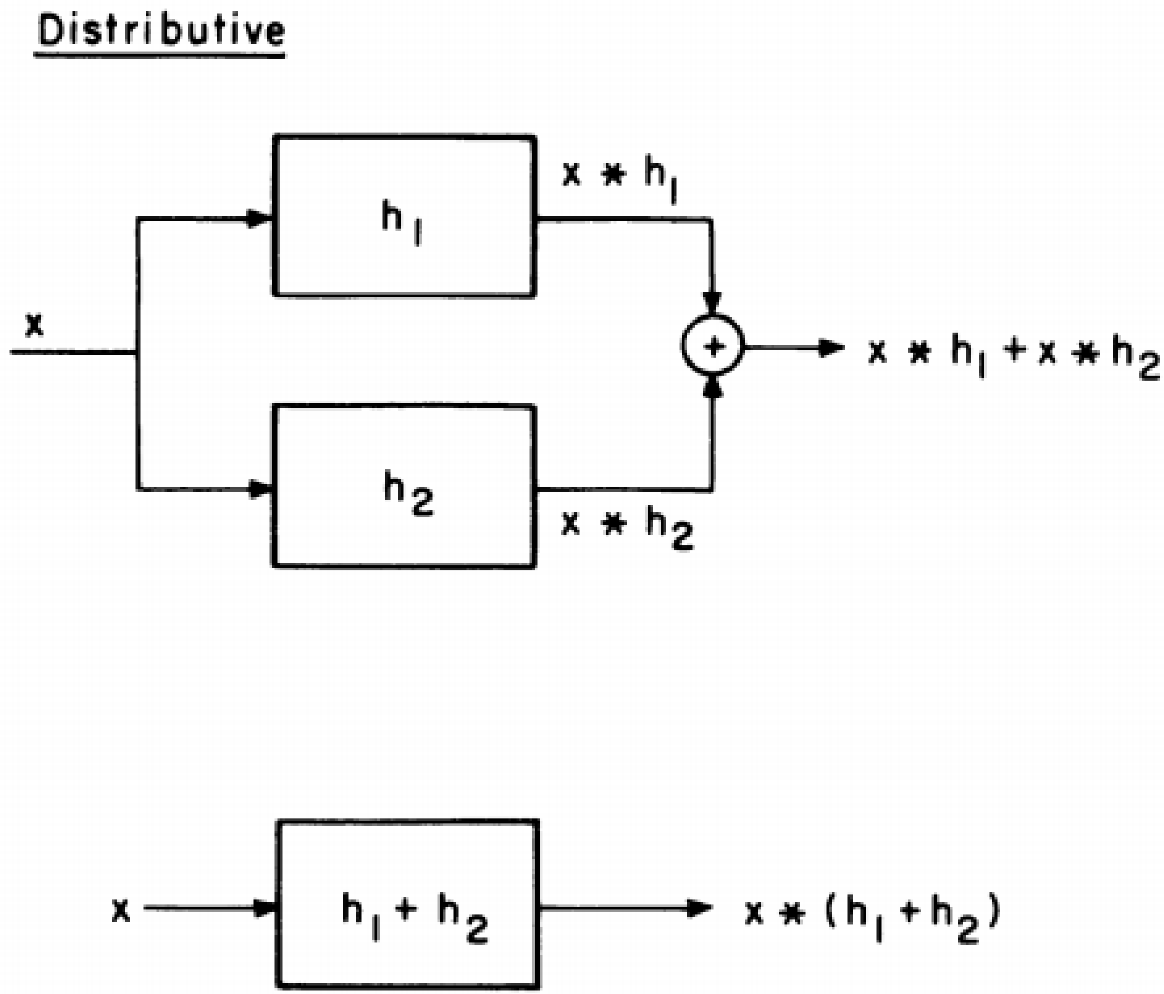


Properties Of LTI system





Properties Of LTI Systems



UNIT-II
Analysis of Continuous-Time Signals

Continuous-Time Sinusoidal signal

$$g(t) = A\cos(2\pi t / T_0 + \theta) = A\cos(2\pi f_0 t + \theta) = A\cos(\omega_0 t + \theta)$$

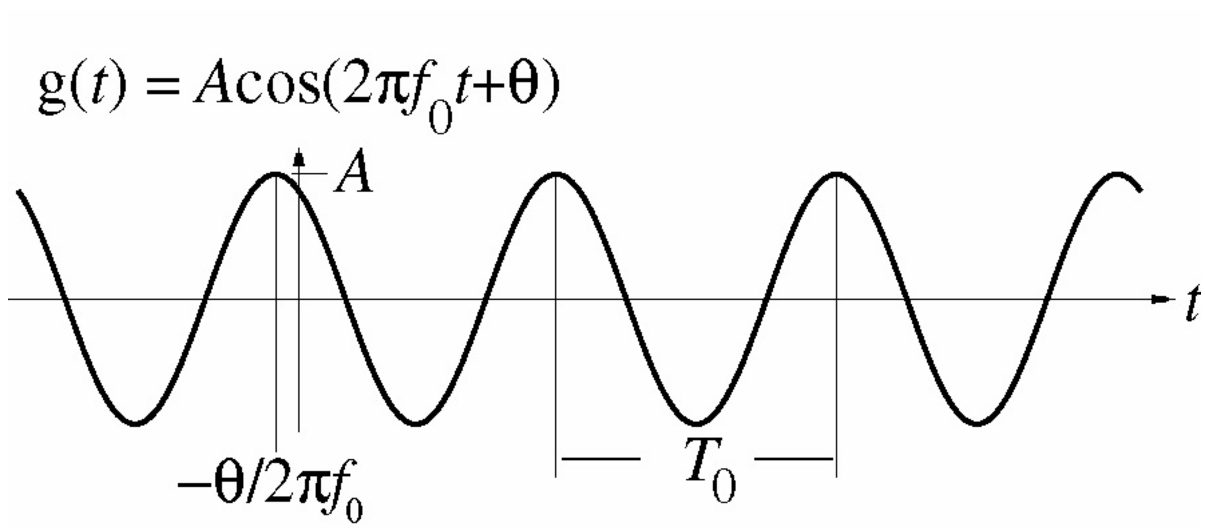
Amplitude

Period (s)

Phase Shift (radians)

Cyclic Frequency (Hz)

Radian Frequency (radians/s)

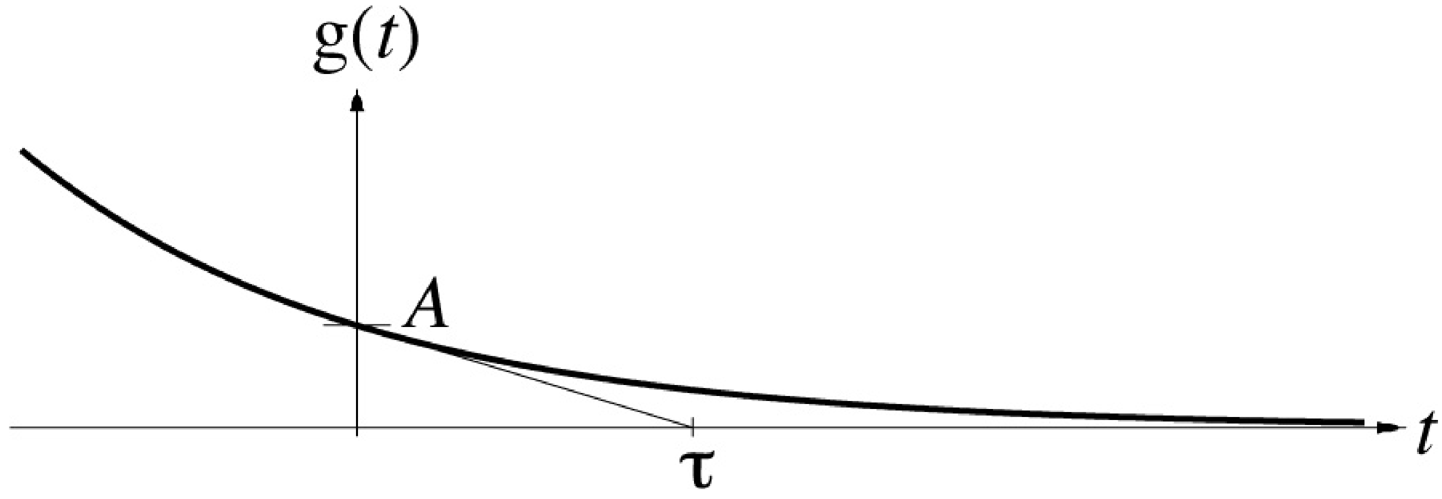


•*Continuous-Time Exponentials*

$$g(t) = Ae^{-t/\tau}$$

$$\uparrow \quad \uparrow$$

Amplitude Time Constant (s)



•*Exponential Fourier Series*

The Fourier series representation of a signal $\mathrm{x}(t)$ over a time $t_0 < t < t_0 + T$ is

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi kt/T}$$

where $\mathrm{c_x}[k]$ is the **harmonic function** and k is the **harmonic number**. The harmonic function can be found from the signal using the princple of **orthogonality**.

$$\mathrm{x}(t) = \sum_{k=-\infty}^{\infty} \mathrm{c_x}[k] e^{j2\pi kt/T} \quad \text{and} \quad \mathrm{c_x}[k] = \frac{1}{T} \int_{t_0}^{t_0+T} \mathrm{x}(t) e^{-j2\pi kt/T}\, dt.$$

The signal and its harmonic function form a **Fourier series pair** $\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k]$ where T is the representation time and, therefore, the fundamental period of the CTFS representation of $\mathrm{x}(t)$ If T is also period of $\mathrm{x}(t)$, the CTFS representation of $\mathrm{x}(t)$ is valid for all time. This is, by far, the most common use of the CTFS in engineering applications. If T is not a period of $\mathrm{x}(t)$, the CTFS representation is generally valid only in the interval $t_0 \le t < t_0 + T$.

Trignometric CTFS

For a real valued function x(t),

$$\mathrm{c_x}[k] = \mathrm{c_x^*}[-k]$$

$$\mathrm{x}(t) = \mathrm{a_x}[0] + \sum_{k=1}^{\infty} \left\{ \mathrm{a_x}[k]\cos(2\pi kt/T) + \mathrm{b_x}[k]\sin(2\pi kt/T) \right\}$$

where

$$\mathrm{a_x}[k] = \frac{2}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t)\cos(2\pi kt/T)\,dt$$

$$\mathrm{b_x}[k] = \frac{2}{T}\int_{t_0}^{t_0+T} \mathrm{x}(t)\sin(2\pi kt/T)\,dt$$

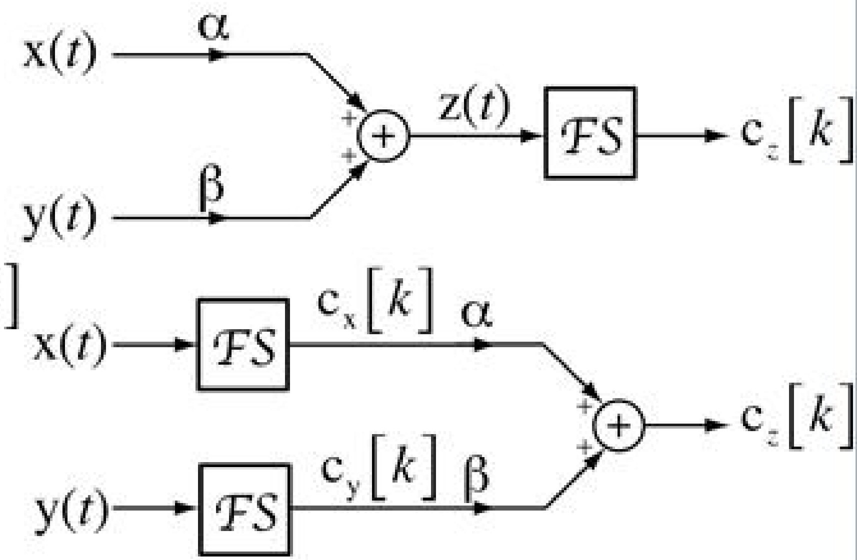
Trignometric CTFS

For an **even function**, the complex CTFS harmonic function $c_x[k]$ is **purely real** and the sine harmonic function $a_x[k]$ is zero.

For an **odd function**, the complex CTFS harmonic function $c_x[k]$ is **purely imaginary** and the cosine harmonic function $b_x[k]$ is zero.

CTFS Properties

Let a signal $x(t)$ have a fundamental period T_{0x} and let a signal $y(t)$ have a fundamental period T_{0y}. Let the CTFS harmonic functions, each using a common period T as the representation time, be $c_x[k]$ and $c_y[k]$. Then the following properties apply.

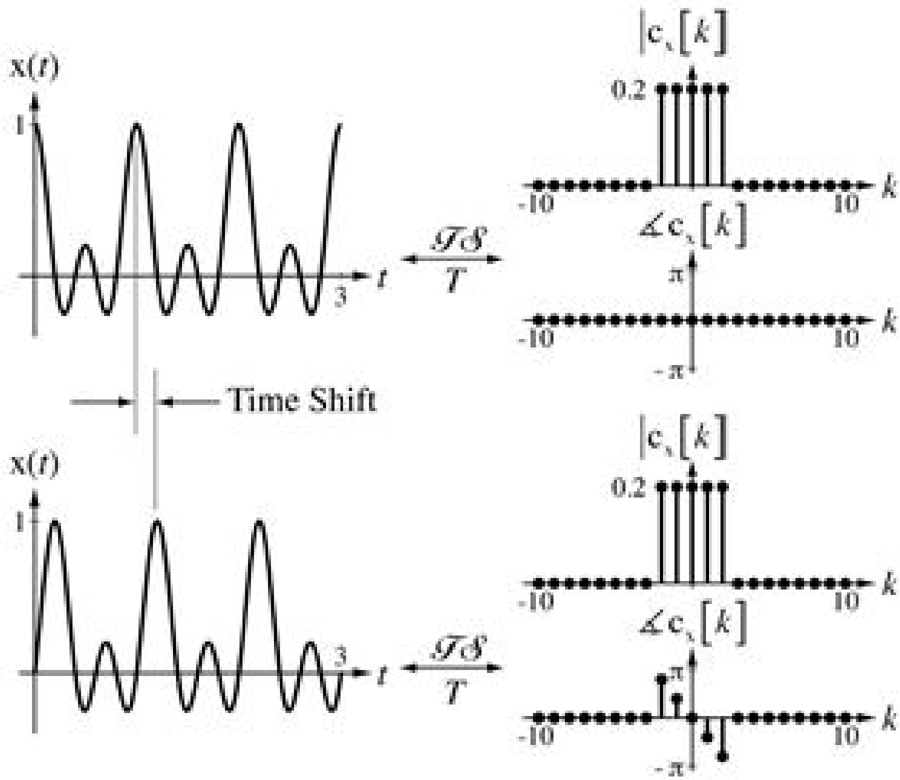


Linearity

$$\alpha\,x(t)+\beta\,y(t) \xleftrightarrow[T]{FS} \alpha\,c_x[k]+\beta\,c_y[k]$$

CTFS Properties

Time Shifting $\quad \mathrm{x}(t-t_0) \xleftrightarrow[T]{\mathcal{FS}} e^{-j2\pi k t_0/T}\, \mathrm{c_x}[k]$



CTFS Properties

Frequency Shifting (Harmonic Number Shifting)

$$e^{j2\pi k_0 t/T}\,\mathrm{x}(t) \xleftrightarrow[T]{\mathcal{F}S} \mathrm{c_x}[k-k_0]$$

A shift in frequency (harmonic number) corresponds to multiplication of the time function by a complex exponential.

Time Reversal

$$\mathrm{x}(-t) \xleftrightarrow[T]{\mathcal{F}S} \mathrm{c_x}[-k]$$

CTFS Properties

Time Scaling

Let $z(t) = x(at)$, $a > 0$

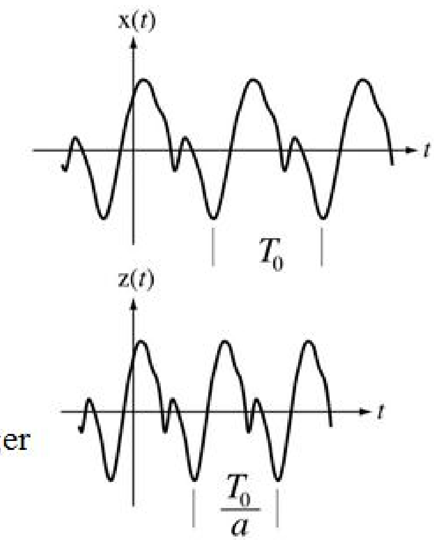
Case 1. $T = T_{0x}/a = T_{0z}$ for $z(t)$

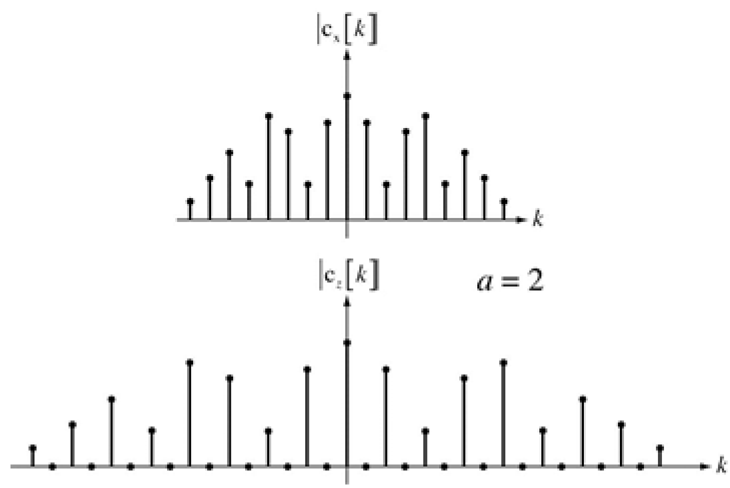
$$c_z[k] = c_x[k]$$

Case 2. $T = T_{0x}$ for $z(t)$

If a is an integer,

$$c_z[k] = \begin{cases} c_x[k/a] & , \ k/a \text{ an integer} \\ 0 & , \ \text{otherwise} \end{cases}$$

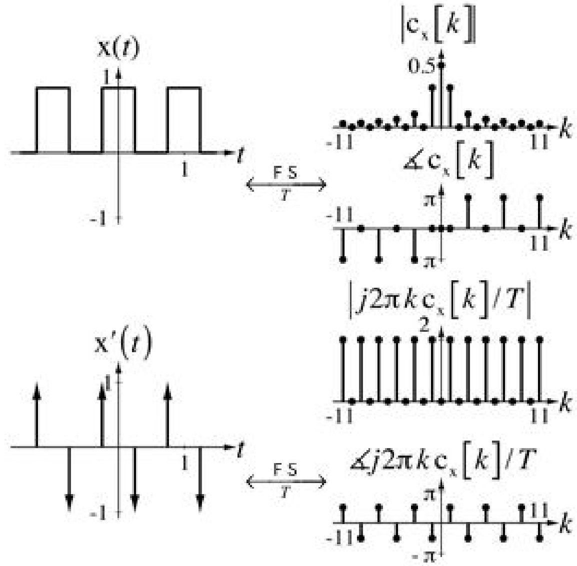




CTFS Properties

Time Differentiation

$$\frac{d}{dt}\left(\mathrm{x}(t)\right) \xleftrightarrow[T]{\mathcal{F}S} j2\pi k\,\mathrm{c_x}[k]/T$$



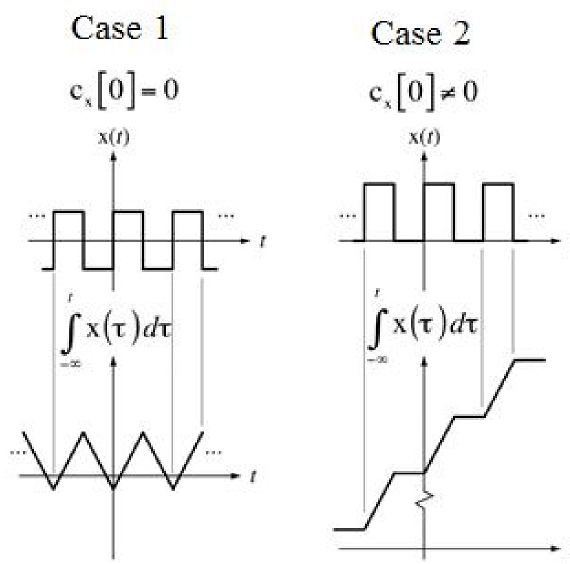
Time Integration

Case 1. $\mathrm{c_x}[0] = 0$

$$\int_{-\infty}^{t} \mathrm{x}(\lambda)\,d\lambda \xleftrightarrow[T]{\mathcal{F}S} \frac{\mathrm{c_x}[k]}{j2\pi k/T}\quad , k \neq 0$$

Case 2. $\mathrm{c_x}[0] \neq 0$

$\int_{-\infty}^{t} \mathrm{x}(\lambda)\,d\lambda$ is not periodic



CTFS Properties

Multiplication - Convolution Duality

$$\mathrm{x}(t)\mathrm{y}(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c_x}[k] * \mathrm{c_y}[k]$$

(The harmonic functions $\mathrm{c_x}[k]$ and $\mathrm{c_y}[k]$ must be based on the same representation time T.)

$$\mathrm{x}(t) \circledast \mathrm{y}(t) \xleftrightarrow[T]{\mathcal{FS}} T\,\mathrm{c_x}[k]\mathrm{c_y}[k]$$

The symbol $\circledast$ indicates **periodic convolution**.

Periodic convolution is defined mathematically by

$$\mathrm{x}(t) \circledast \mathrm{y}(t) = \int_T \mathrm{x}(\tau)\mathrm{y}(t-\tau)\,d\tau$$

$\mathrm{x}(t) \circledast \mathrm{y}(t) = \mathrm{x}_{ap}(t) * \mathrm{y}(t)$ where $\mathrm{x}_{ap}(t)$ is any single period of $\mathrm{x}(t)$

Conjugation

$$\mathrm{x}^*(t) \xleftrightarrow[T]{\mathcal{FS}} \mathrm{c}_{\mathrm{x}}^*[-k]$$

Parseval's Theorem

$$\frac{1}{T}\int_T |\mathrm{x}(t)|^2\,dt = \sum_{k=-\infty}^{\infty} |\mathrm{c_x}[k]|^2$$

The **average power** of a periodic signal is the sum of the average powers in its harmonic components.

Continuous Time Fourier Transforms

Forward $\boxed{f \text{ form}}$ Inverse

$$\mathrm{X}(f) = \mathcal{F}\left(\mathrm{x}(t)\right) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j2\pi f t}\, dt \qquad \mathrm{x}(t) = \mathcal{F}^{-1}\left(\mathrm{X}(f)\right) = \int_{-\infty}^{\infty} \mathrm{X}(f) e^{+j2\pi f t}\, df$$

Forward $\boxed{\omega \text{ form}}$ Inverse

$$\mathrm{X}(j\omega) = \mathcal{F}\left(\mathrm{x}(t)\right) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-j\omega t}\, dt \qquad \mathrm{x}(t) = \mathcal{F}^{-1}\left(\mathrm{X}(j\omega)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{X}(j\omega) e^{+j\omega t}\, d\omega$$

Commonly-used notation:

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(f) \quad \text{or} \quad \mathrm{x}(t) \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)$$

Some CTFT Pairs

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1$$

$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha) \;,\; \alpha>0$$

$$-e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha) \;,\; \alpha<0$$

$$te^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha)^2 \;,\; \alpha>0$$

$$-te^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} 1/(j\omega+\alpha)^2 \;,\; \alpha<0$$

$$t^n e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{n!}{(j\omega+\alpha)^{n+1}} \;,\; \alpha>0$$

$$-t^n e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{n!}{(j\omega+\alpha)^{n+1}} \;,\; \alpha<0$$

$$e^{-\alpha t}\sin(\omega_0 t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha>0$$

$$-e^{-\alpha t}\sin(\omega_0 t)\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{\omega_0}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha<0$$

$$e^{-\alpha t}\cos(\omega_0 t)\,\mathrm{u}(t) \xleftrightarrow{\mathcal{F}} \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha>0$$

$$-e^{-\alpha t}\cos(\omega_0 t)\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{F}} \frac{j\omega+\alpha}{(j\omega+\alpha)^2+\omega_0^2} \;,\; \alpha<0$$

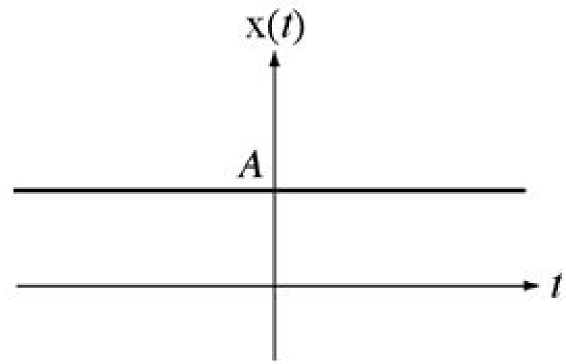
$$e^{-\alpha|t|} \xleftrightarrow{\mathcal{F}} \frac{2\alpha}{\omega^2+\alpha^2} \;,\; \alpha>0$$

Convergence and the Generalized Fourier Transform

Let $x(t) = A$. Then from the definition of the CTFT,

$$X(f) = \int_{-\infty}^{\infty} A e^{-j2\pi f t}\, dt = A \int_{-\infty}^{\infty} e^{-j2\pi f t}\, dt$$

This integral does not converge so, strictly speaking, the CTFT does not exist.



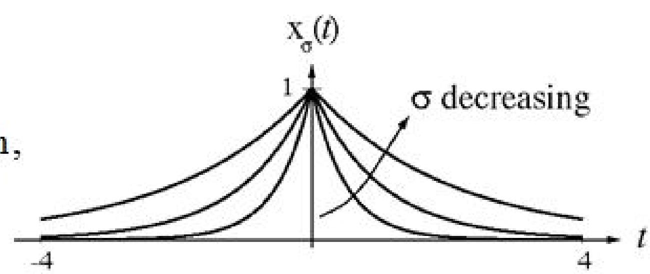
But consider a similar function,

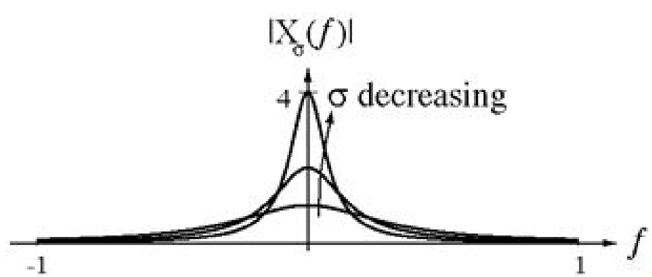
$$x_\sigma(t) = A e^{-\sigma|t|}\ , \quad \sigma > 0$$

Its CTFT integral

$$X_\sigma(f) = \int_{-\infty}^{\infty} A e^{-\sigma|t|} e^{-j2\pi f t}\, dt$$

does converge.





Convergence and the Generalized Fourier Transform

Carrying out the integral, $\mathrm{X}_\sigma(f) = A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2}$.

Now let σ approach zero.

If $f \neq 0$ then $\lim\limits_{\sigma \to 0} A\dfrac{2\sigma}{\sigma^2 + (2\pi f)^2} = 0$. The area under this function is $A\displaystyle\int_{-\infty}^{\infty} \dfrac{2\sigma}{\sigma^2 + (2\pi f)^2}\,df$ which is A, independent of the value of σ. So, in the limit as σ approaches zero, the CTFT has an area of A and is zero unless $f = 0$. This exactly defines an **impulse** of strength A. Therefore $A \xleftrightarrow{\mathcal{F}} A\delta(f)$.

By a similar process it can be shown that

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$$

and

$$\sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{j}{2}\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

More CTFT Pairs

The generalization of the CTFT allows us to extend the table of CTFT pairs to some very useful functions.

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \qquad 1 \xleftrightarrow{\mathcal{F}} \delta(f)$$

$$\mathrm{sgn}(t) \xleftrightarrow{\mathcal{F}} 1/j\pi f \qquad \mathrm{u}(t) \xleftrightarrow{\mathcal{F}} (1/2)\delta(f) + 1/j2\pi f$$

$$\mathrm{rect}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}(f) \qquad \mathrm{sinc}(t) \xleftrightarrow{\mathcal{F}} \mathrm{rect}(f)$$

$$\mathrm{tri}(t) \xleftrightarrow{\mathcal{F}} \mathrm{sinc}^2(f) \qquad \mathrm{sinc}^2(t) \xleftrightarrow{\mathcal{F}} \mathrm{tri}(f)$$

$$\delta_{T_0}(t) \xleftrightarrow{\mathcal{F}} f_0 \delta_{f_0}(f)\ ,\ f_0 = 1/T_0 \qquad T_0 \delta_{T_0}(t) \xleftrightarrow{\mathcal{F}} \delta_{f_0}(f)\ ,\ T_0 = 1/f_0$$

$$\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} (1/2)\left[\delta(f - f_0) + \delta(f + f_0)\right] \qquad \sin(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} (j/2)\left[\delta(f + f_0) - \delta(f - f_0)\right]$$

CTFT Properties

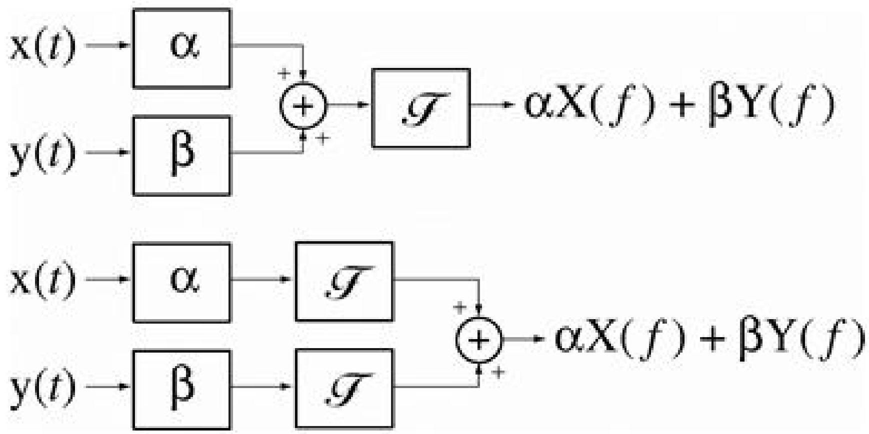
If $\mathsf{F}\left(\mathrm{x}(t)\right)=\mathrm{X}(f)$ or $\mathrm{X}(j\omega)$ and $\mathsf{F}\left(\mathrm{y}(t)\right)=\mathrm{Y}(f)$ or $\mathrm{Y}(j\omega)$

then the following properties can be proven.

Linearity

$$\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t)\xleftrightarrow{\mathsf{F}}\alpha\,\mathrm{X}(f)+\beta\,\mathrm{Y}(f)$$

$$\alpha\,\mathrm{x}(t)+\beta\,\mathrm{y}(t)\xleftrightarrow{\mathsf{F}}\alpha\,\mathrm{X}(j\omega)+\beta\,\mathrm{Y}(j\omega)$$

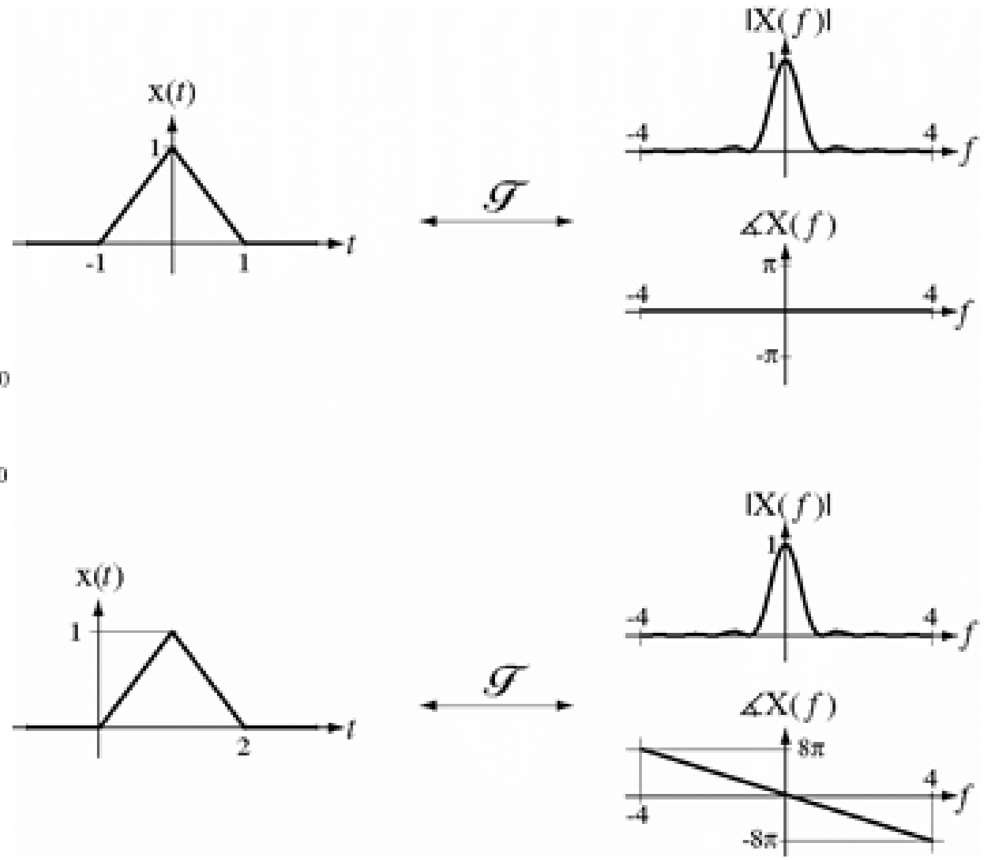


CTFT Properties

Time Shifting

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(f)e^{-j2\pi f t_0}$$

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(j\omega)e^{-j\omega t_0}$$

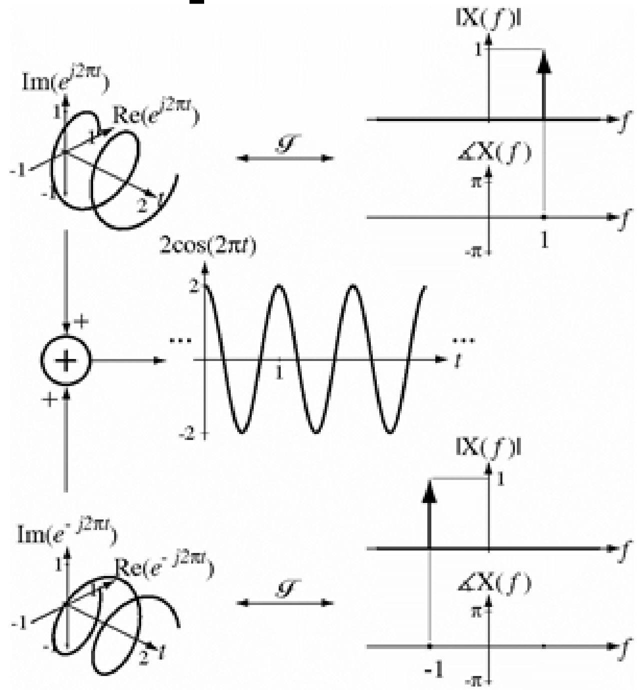


CTFT Properties

Frequency Shifting

$$x(t)e^{+j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} X(f - f_0)$$

$$x(t)e^{+j\omega_0 t} \xleftrightarrow{\mathcal{F}} X(\omega - \omega_0)$$



Time Scaling

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(j\frac{\omega}{a}\right)$$

Frequency Scaling

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(af)$$

$$\frac{1}{|a|} x\left(\frac{t}{a}\right) \xleftrightarrow{\mathcal{F}} X(ja\omega)$$

CTFT Properties

Transform of a Conjugate

$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-f)$$
$$x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

Multiplication Convolution Duality

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(f)Y(f)$$
$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(j\omega)Y(j\omega)$$
$$x(t)y(t) \xleftrightarrow{\mathcal{F}} X(f) * Y(f)$$
$$x(t)y(t) \xleftrightarrow{\mathcal{F}} (1/2\pi)X(j\omega) * Y(j\omega)$$

In the frequency domain, the **cascade connection** multiplies the frequency responses instead of convolving the impulse responses.

$$x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = h(t) * x(t) \qquad X(f) \rightarrow \boxed{H(f)} \rightarrow Y(f) = H(f)X(f)$$

$$X(f) \rightarrow \boxed{H_1(f)} \rightarrow X(f)H_1(f) \rightarrow \boxed{H_2(f)} \rightarrow Y(f) = X(f)H_1(f)H_2(f)$$

$$X(f) \rightarrow \boxed{H_1(f)H_2(f)} \rightarrow Y(f)$$

CTFT Properties

Time Differentiation	$\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j2\pi f\,\mathrm{X}(f)$ $\frac{d}{dt}(\mathrm{x}(t)) \xleftrightarrow{\mathcal{F}} j\omega\,\mathrm{X}(j\omega)$
Modulation	$\mathrm{x}(t)\cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{X}(f-f_0)+\mathrm{X}(f+f_0)\right]$ $\mathrm{x}(t)\cos(\omega_0 t) \xleftrightarrow{\mathcal{F}} \frac{1}{2}\left[\mathrm{X}(j(\omega-\omega_0))+\mathrm{X}(j(\omega+\omega_0))\right]$
Transforms of Periodic Signals	$\mathrm{x}(t)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{-j2\pi k f_F t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(f)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]\delta(f-kf_0)$ $\mathrm{x}(t)=\sum_{k=-\infty}^{\infty}\mathrm{X}[k]e^{-jk\omega_F t} \xleftrightarrow{\mathcal{F}} \mathrm{X}(j\omega)=2\pi\sum_{k=-\infty}^{\infty}\mathrm{X}[k]\delta(\omega-k\omega_0)$
Parseval's Theorem	$\int_{-\infty}^{\infty}\lvert\mathrm{x}(t)\rvert^2\,dt=\int_{-\infty}^{\infty}\lvert\mathrm{X}(f)\rvert^2\,df$ $\int_{-\infty}^{\infty}\lvert\mathrm{x}(t)\rvert^2\,dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}\lvert\mathrm{X}(j\omega)\rvert^2\,df$
Integral Definition of an Impulse	$\int_{-\infty}^{\infty}e^{-j2\pi xy}\,dy=\delta(x)$
Duality	$\mathrm{X}(t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(-f)$ and $\mathrm{X}(-t) \xleftrightarrow{\mathcal{F}} \mathrm{x}(f)$ $\mathrm{X}(jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(-\omega)$ and $\mathrm{X}(-jt) \xleftrightarrow{\mathcal{F}} 2\pi\,\mathrm{x}(\omega)$

CTFT Properties

Total - Area Integral

$$\mathrm{X}(0)=\left[\int_{-\infty}^{\infty}\mathrm{x}(t)e^{-j2\pi ft}\,dt\right]_{f\to 0}=\int_{-\infty}^{\infty}\mathrm{x}(t)\,dt$$

$$\mathrm{x}(0)=\left[\int_{-\infty}^{\infty}\mathrm{X}(f)e^{+j2\pi ft}\,df\right]_{t\to 0}=\int_{-\infty}^{\infty}\mathrm{X}(f)\,df$$

$$\mathrm{X}(0)=\left[\int_{-\infty}^{\infty}\mathrm{x}(t)e^{-j\omega t}\,dt\right]_{\omega\to 0}=\int_{-\infty}^{\infty}\mathrm{x}(t)\,dt$$

$$\mathrm{x}(0)=\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{X}(j\omega)e^{+j\omega t}\,d\omega\right]_{t\to 0}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{X}(j\omega)\,d\omega$$

Integration

$$\int_{-\infty}^{t}\mathrm{x}(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(f)}{j2\pi f}+\frac{1}{2}\mathrm{X}(0)\delta(f)$$

$$\int_{-\infty}^{t}\mathrm{x}(\lambda)\,d\lambda \xleftrightarrow{\mathcal{F}} \frac{\mathrm{X}(j\omega)}{j\omega}+\pi\,\mathrm{X}(0)\delta(\omega)$$

Fourier Transform Examples

Impulses

(a) $x(t) = \delta(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

$$\Downarrow$$

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{j\omega t} d\omega$$

(b) $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt$$

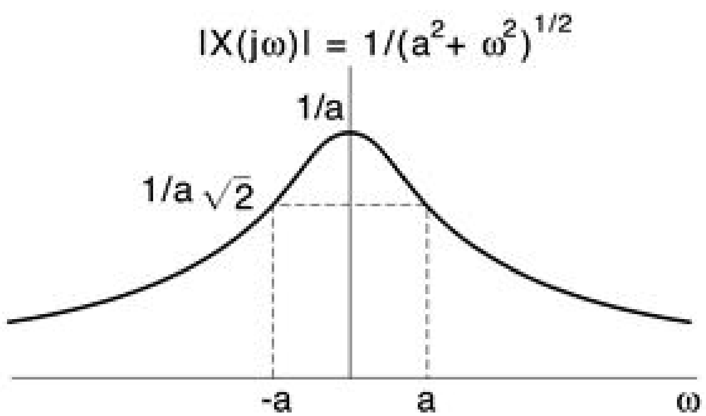
$$= e^{-j\omega t_0}$$

Fourier Transform of a Right sided Exponential

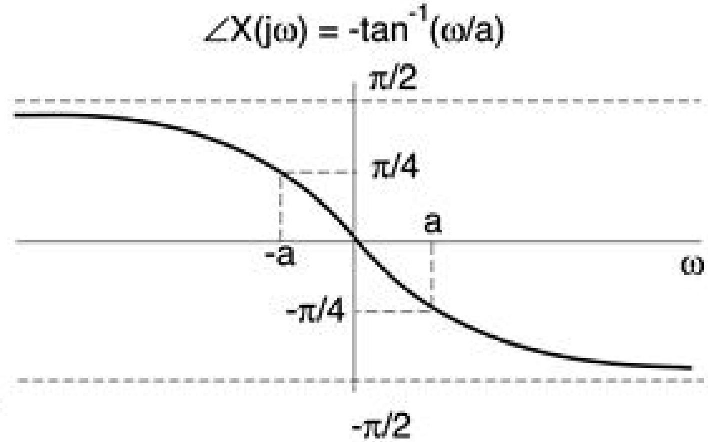
$$x(t) = e^{-at}u(t), a > 0$$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{0}^{\infty} \underbrace{e^{-at}e^{-j\omega t}}_{e^{-(a+j\omega)t}}dt$$

$$= -\left(\frac{1}{a+j\omega}\right) e^{-(a+j\omega)t}\Big|_{0}^{\infty} = \frac{1}{a+j\omega}$$



Even symmetry



Odd symmetry

CT Fourier Transforms of Periodic Signals

Suppose

$$X(j\omega) = \delta(\omega - \omega_0)$$

$$\Downarrow$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{j\omega t} d\omega = \frac{1}{2\pi} e^{j\omega_0 t}$$

That is

$$\boxed{e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)}$$

More generally

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Laplace Transform

The CTFT expresses a time-domain signal as a linear combination of **complex sinusoids** of the form $e^{j\omega t}$. In the generalization of the CTFT to the Laplace transform, the complex sinusoids become **complex exponentials** of the form e^{st} where s can have any complex value. Replacing the complex sinusoids with complex exponentials leads to this definition of the Laplace transform.

$$\mathcal{L}\left(\mathrm{x}(t)\right) = \mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-st}\, dt$$

$$\mathrm{x}(t) \xleftrightarrow{\mathcal{L}} \mathrm{X}(s)$$

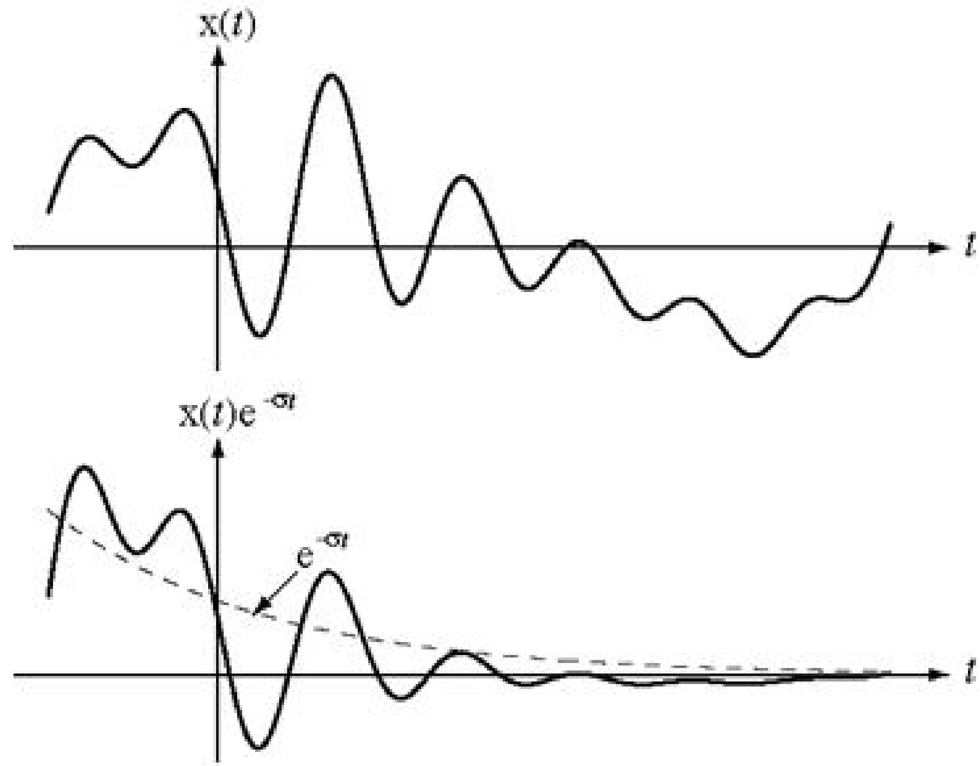
Generalizing the Fourier Transform

The variable s is viewed as a generalization of the variable ω of the form $s = \sigma + j\omega$. Then, when σ, the real part of s, is zero, the Laplace transform reduces to the CTFT. Using $s = \sigma + j\omega$ the Laplace transform is

$$\mathrm{X}(s) = \int_{-\infty}^{\infty} \mathrm{x}(t) e^{-(\sigma + j\omega)t}\, dt$$

$$= \mathcal{F}\left[\mathrm{x}(t) e^{-\sigma t}\right]$$

which is the Fourier transform of $\mathrm{x}(t) e^{-\sigma t}$



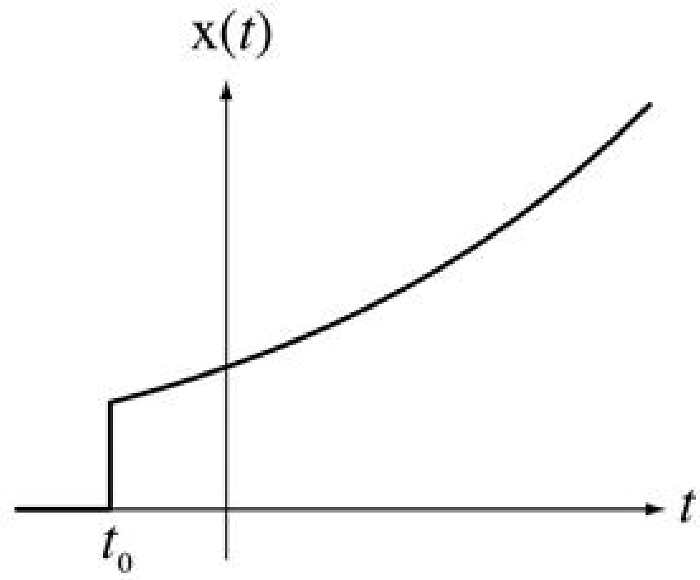
Existence of Laplace Transform

Right-Sided Exponential

$$\mathrm{x}(t) = e^{\alpha t}\,\mathrm{u}(t - t_0) \quad , \quad \alpha \in \mathbb{R}$$

$$\mathrm{X}(s) = \int_{t_0}^{\infty} e^{\alpha t} e^{-st}\, dt = \int_{t_0}^{\infty} e^{(\alpha-\sigma)t} e^{-j\omega t}\, dt$$

If $\mathrm{Re}(s) = \sigma > \alpha$ the asymptotic behavior of $e^{(\alpha-\sigma)t} e^{-j\omega t}$ as $t \to \infty$ is to approach zero and the Laplace transform integral converges.



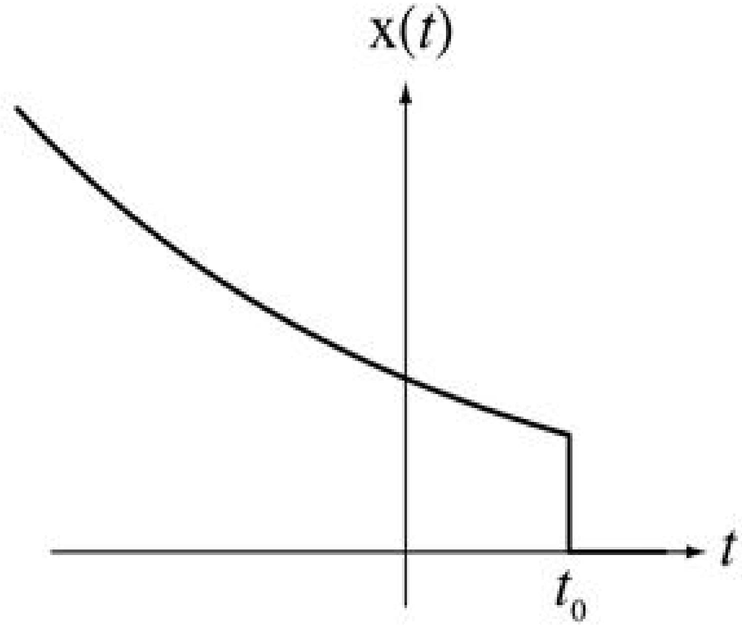
Existence of Laplace Transform

Left-Sided Exponential

$$x(t) = e^{\beta t}\, u(t_0 - t) \quad , \quad \beta \in \mathbb{R}$$

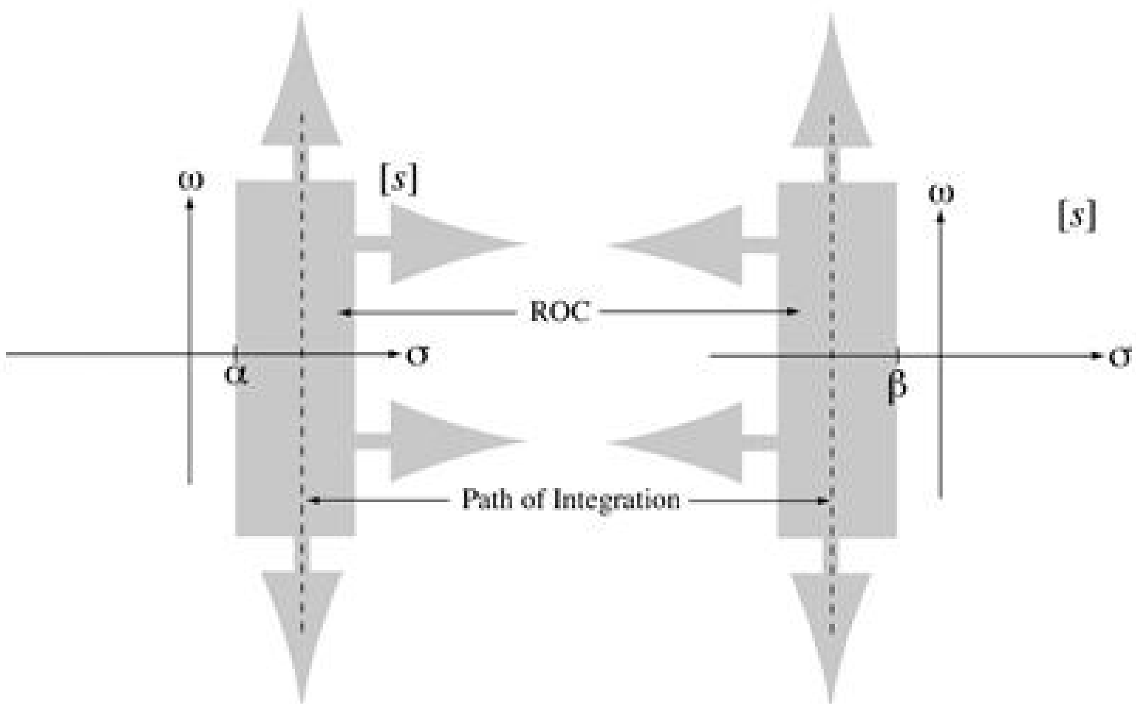
$$X(s) = \int_{-\infty}^{t_0} e^{\beta t} e^{-st}\, dt = \int_{-\infty}^{t_0} e^{(\beta-\sigma)t} e^{-j\omega t}\, dt$$

If $\sigma < \beta$ the asymptotic behavior of $e^{(\beta-\sigma)t} e^{-j\omega t}$ as $t \to -\infty$ is to approach zero and the Laplace transform integral converges.



Existence of Laplace Transform

The two conditions $\sigma > \alpha$ and $\sigma < \beta$ define the **region of convergence** (**ROC**) for the Laplace transform of right- and left-sided signals.



Region of Convergence

The following two Laplace transform pairs illustrate the importance of the region of convergence.

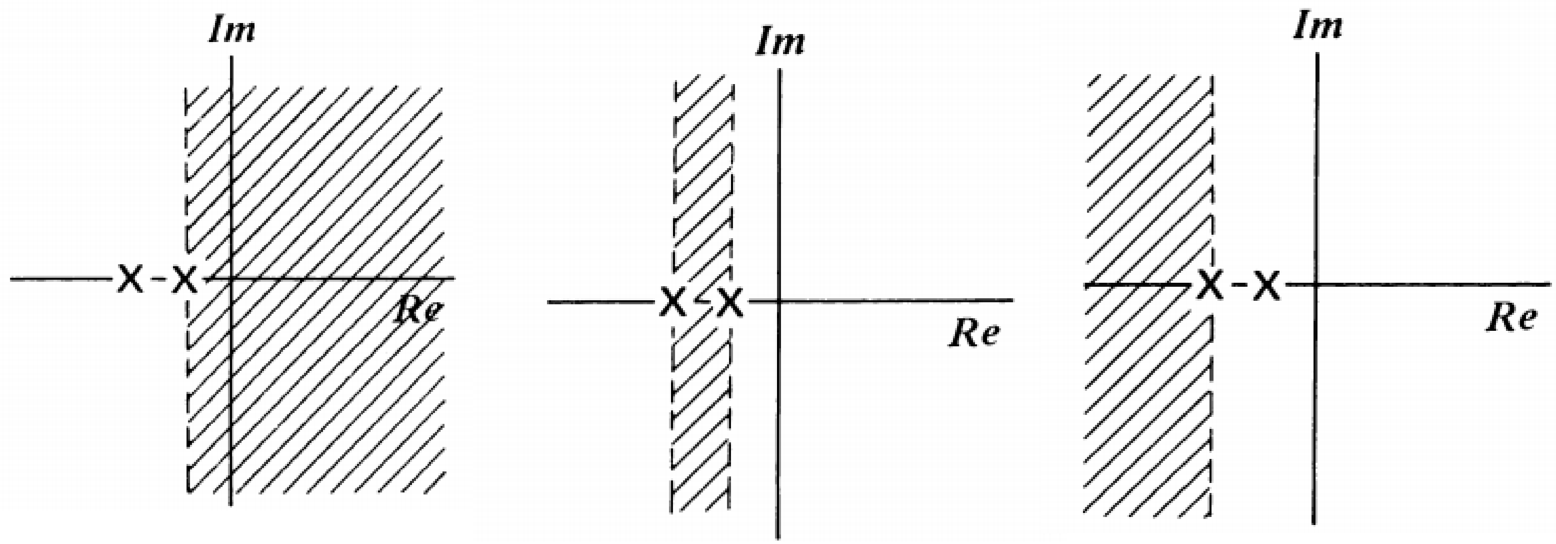
$$e^{-\alpha t}\,\mathrm{u}(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \quad , \quad \sigma > -\alpha$$

$$-\,e^{-\alpha t}\,\mathrm{u}(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+\alpha} \quad , \quad \sigma < -\alpha$$

The two time-domain functions are different but the algebraic expressions for their Laplace transforms are the same. Only the ROC's are different.

Importance of ROC

$$X(s) = \frac{1}{(s+1)\,(s+2)} \qquad X(s) = \frac{1}{(s+1)\,(s+2)} \qquad X(s) = \frac{1}{(s+1)\,(s+2)}$$



Properties of ROC

PROPERTIES OF THE REGION OF CONVERGENCE

- The ROC contains no poles

$$X(s) = \frac{N(s)}{D(s)}$$

poles of $X(s) \Rightarrow D(s) = 0$

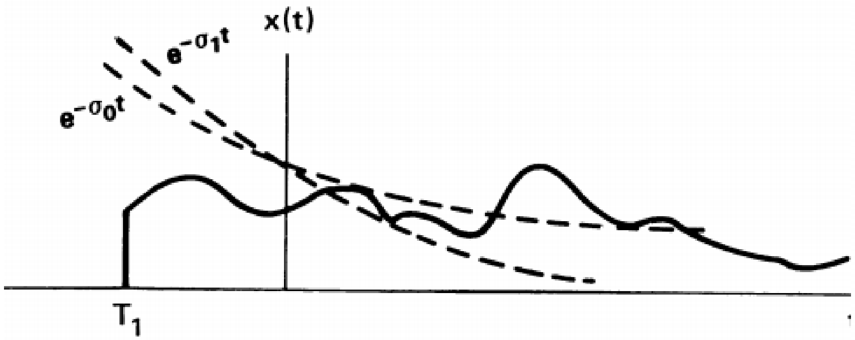
- The ROC of $X(s)$ consists of a strip parallel to the $j\omega$-axis in the s-plane

- $\mathcal{F}\{x(t)\}$ converges $\Leftrightarrow$ ROC includes the $j\omega$-axis in the s-plane

Region of Convergence

x(t) right-sided and $Re\{s\} = \sigma_o$ is in ROC

=> all values for which $Re\{s\} > \sigma_o$ are in ROC



x(t) right-sided and X(s) rational

=> ROC is to the right of the rightmost pole.

- x(t) left-sided and $Re\{s\} = \sigma_o$ is in ROC
  => all values for which $Re\{s\} < \sigma_o$ are in ROC

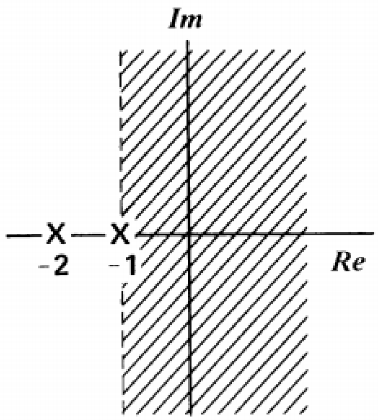
- x(t) left-sided and X(s) rational
  => ROC to the left of the leftmost pole.

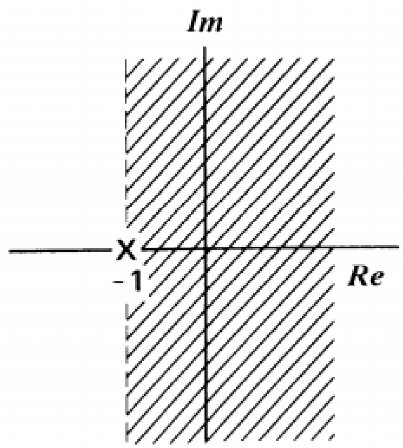
- x(t) two-sided and $Re\{s\} = \sigma_o$ is in ROC
  => ROC is a strip in the s-plane

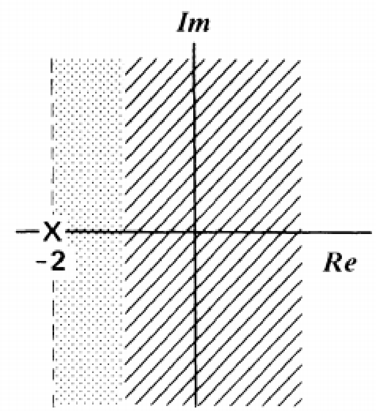
Inverse Laplace Transform

- Decomposing a specified Laplace transform into a partial fraction expansion.

Find the inverse Laplace Transform gi $X(s) = \frac{1}{(s+1)\,(s+2)}$







$$X(s) = \frac{1}{(s+1)\,(s+2)} \qquad \mathcal{Re}\{s\} > -1$$

$$= \frac{1}{s+1} - \frac{1}{s+2} \qquad \mathcal{Re}\{s\} > -1$$

$$X_1(s) = \frac{1}{s+1} \qquad \mathcal{Re}\{s\} > -1$$

$$x_1(t) = e^{-t}\,u(t)$$

$$X_2(s) = \frac{-1}{s+2} \qquad \mathcal{Re}\{s\} > -1$$

$$x_2(t) = -e^{-2t}\,u(t)$$

UNIT III
LTI-CT SYSTEMS

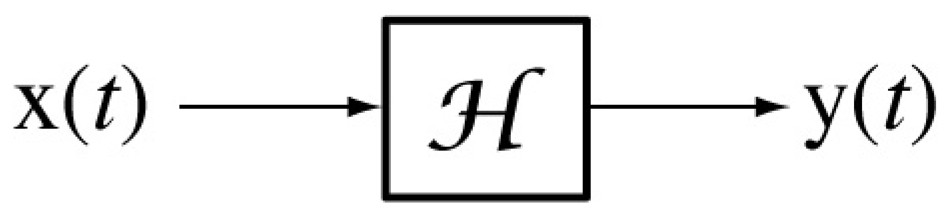
Systems

Systems have **inputs** and **outputs**

Systems accept **excitations** or **input signals** at their inputs and produce **responses** or **output signals** at their outputs

Engineering system analysis is the application of mathematical methods to the design and analysis of systems. Systems are often usefully represented by **block diagrams**

- *A single-input, single-output system block diagram*



Linear Time Invariant Systems

A system satisfying both the **linearity** and the **time-invariance** property.

LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design.

Highly useful signal processing algorithms have been developed utilizing this class of systems over the last several decades.

They possess superposition theorem.

Representation of LTI systems

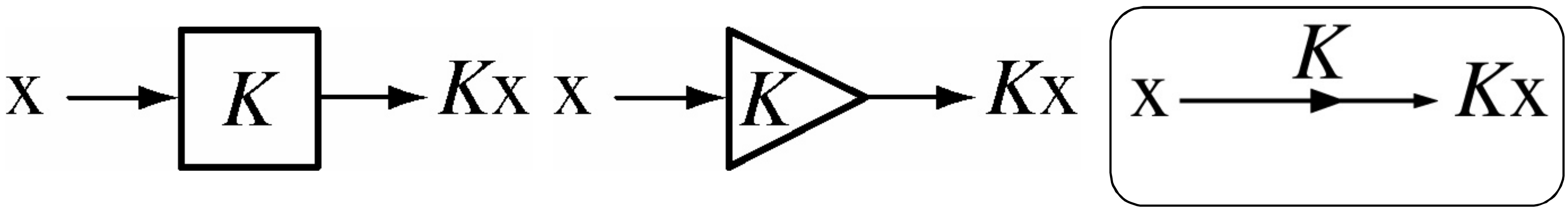
Any linear time-invariant system (LTI) system, continuous-time or discrete-time, can be uniquely characterized by its

- **Impulse response:** response of system to an impulse
- **Frequency response**: response of system to a complex exponential $e^{j 2 \pi f}$ for all possible frequencies f.
- **Transfer function**: Laplace transform of impulse response

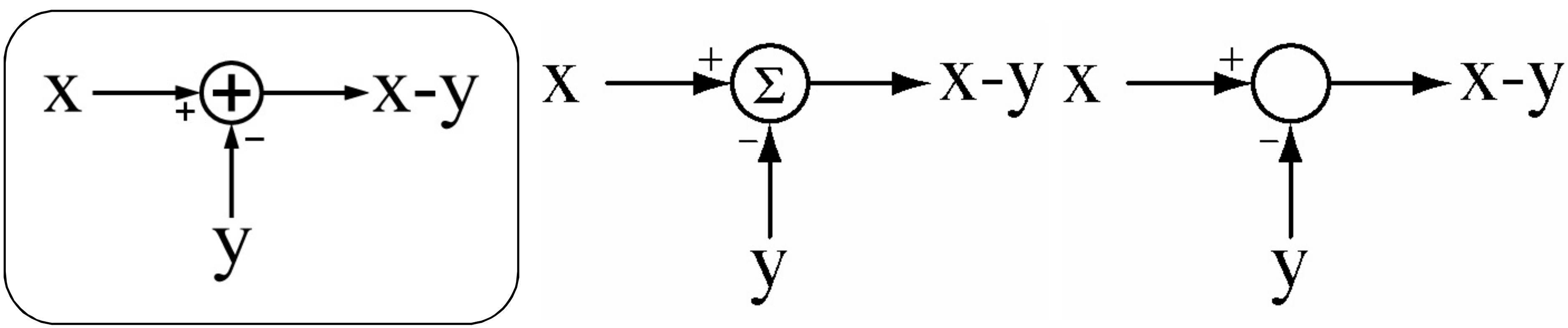
Given one of the three, we can find other two provided that they exist

Block Diagram Symbols

•*Three common block diagram symbols for an **amplifier** (we will*
•*use the last one).*



•*Three common block diagram symbols for a **summing junction***
•*(we will use the first one).*



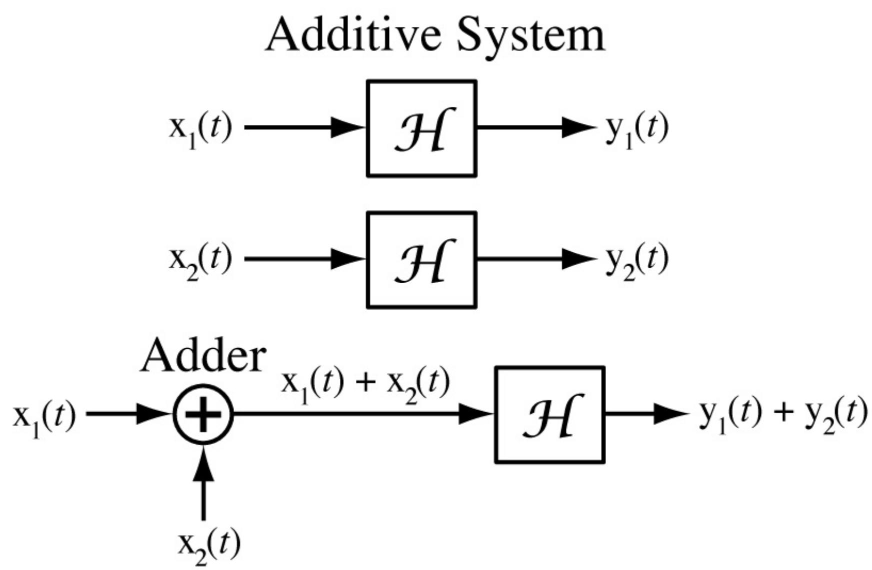
Block Diagram Symbols

- *Block diagram symbol for an* ***integrator***

$$x(t) \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)\,d\tau$$

Additivity

If one excitation causes a zero-state response and another excitation causes another zero-state response and if, for any arbitrary excitations, the sum of the two excitations causes a zero-state response that is the sum of the two zero-state responses, the system is said to be **additive**.



If $g(t) \xrightarrow{H} y_1(t)$ and $h(t) \xrightarrow{H} y_2(t)$

and $g(t) + h(t) \xrightarrow{H} y_1(t) + y_2(t) \Rightarrow H$ is Additive

Convolution Integral

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \longrightarrow \underbrace{y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}_{\text{Convolution Integral}}$$

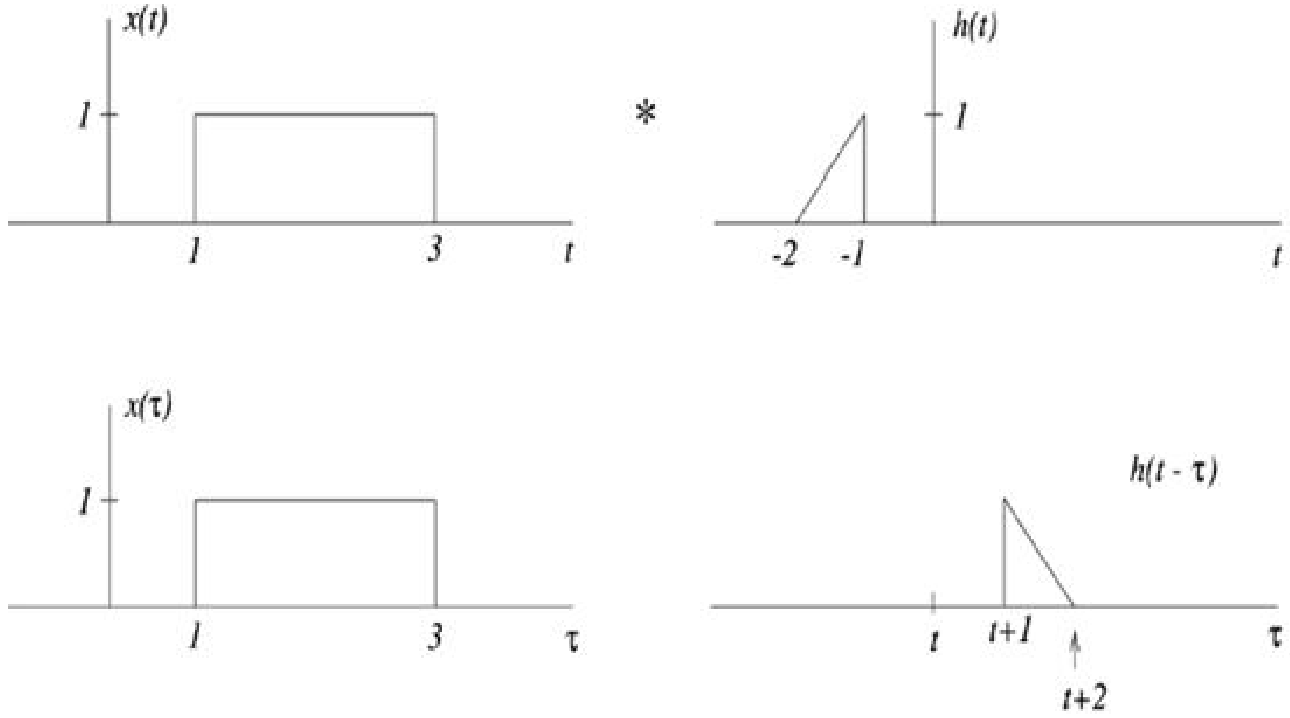
$$\boxed{y(t) = x(t) * h(t) \equiv \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau}$$

$$h(\tau) \xrightarrow{\text{Flip}} h(-\tau) \quad \xrightarrow{\text{Slide}} \quad h(t-\tau)$$

$$\xrightarrow{\text{Multiply}} \quad x(\tau)h(t-\tau) \xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

Convolution Integral

Example



Impulse Response

Let a system be described by

$$a_2\,\mathrm{y}''(t)+a_1\,\mathrm{y}'(t)+a_0\,\mathrm{y}(t)=\mathrm{x}(t)$$

and let the excitation be a unit impulse at time $t=0$. Then the zero-state response y is the impulse response h.

$$a_2\,\mathrm{h}''(t)+a_1\,\mathrm{h}'(t)+a_0\,\mathrm{h}(t)=\delta(t)$$

Since the impulse occurs at time $t=0$ and nothing has excited the system before that time, the impulse response before time $t=0$ is zero (because this is a causal system). After time $t=0$ the impulse has occurred and gone away. Therefore there is no longer an excitation and the impulse response is the homogeneous solution of the differential equation.

Impulse Response

Continuous-time LTI systems are described by differential equations of the general form,

$$a_n \mathrm{y}^{(n)}(t) + a_{n-1} \mathrm{y}^{(n-1)}(t) + \cdots + a_1 \mathrm{y}'(t) + a_0 \mathrm{y}(t)$$
$$= b_m \mathrm{x}^{(m)}(t) + b_{m-1} \mathrm{x}^{(m-1)}(t) + \cdots + b_1 \mathrm{x}'(t) + b_0 \mathrm{x}(t)$$

For all times, $t < 0$:

If the excitation $\mathrm{x}(t)$ is an impulse, then for all time $t < 0$ it is zero. The response $\mathrm{y}(t)$ is zero before time $t = 0$ because there has never been an excitation before that time.

$$a_n \mathrm{y}^{(n)}(t) + a_{n-1} \mathrm{y}^{(n-1)}(t) + \cdots + a_1 \mathrm{y}'(t) + a_0 \mathrm{y}(t)$$
$$= b_m \mathrm{x}^{(m)}(t) + b_{m-1} \mathrm{x}^{(m-1)}(t) + \cdots + b_1 \mathrm{x}'(t) + b_0 \mathrm{x}(t)$$

Case 1: $m < n$

If the response contained an impulse at time $t = 0$ then the nth derivative of the response would contain the nth derivative of an impulse. Since the excitation contains only the mth derivative of an impulse and $m < n$, the differential equation cannot be satisfied at time $t = 0$. Therefore the response cannot contain an impulse or any derivatives of an impulse.

Impulse Response

Case 2: $m = n$

In this case the highest derivative of the excitation and response are the same and the response could contain an impulse at time $t = 0$ but no derivatives of an impulse.

Case 3: $m > n$

In this case, the response could contain an impulse at time $t = 0$ plus derivatives of an impulse up to the $(m - n)$th derivative.

Case 3 is rare in the analysis of practical systems.

Example

To find the constant K integrate $\mathrm{h}'(t)+3\mathrm{h}(t)=\delta(t)$ over the infinitesimal range 0^- to 0^+.

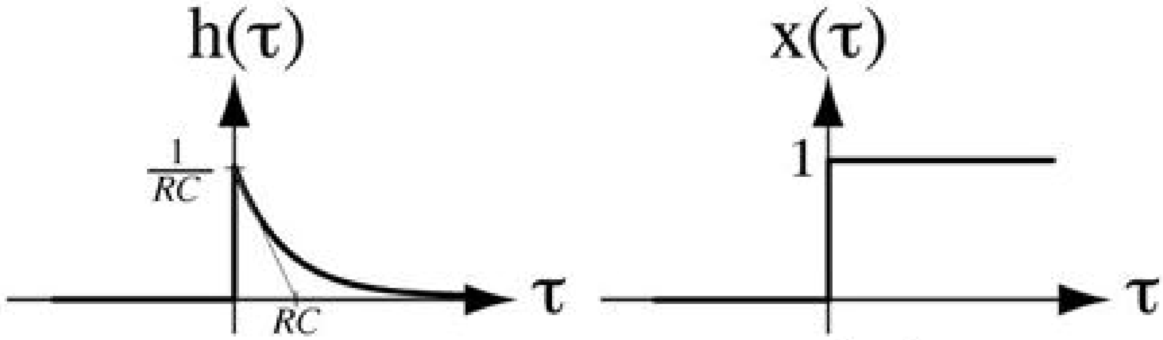
$$\int_{0^-}^{0^+}\mathrm{h}'(t)\,dt+3\int_{0^-}^{0^+}\mathrm{h}(t)=\int_{0^-}^{0^+}\delta(t)$$

$$\underbrace{\mathrm{h}(0^+)}_{=K}-\underbrace{\mathrm{h}(0^-)}_{=0}+3\int_{0^-}^{0^+}Ke^{-3t}\,\mathrm{u}(t)\,dt=\underbrace{\mathrm{u}(0^+)}_{=1}-\underbrace{\mathrm{u}(0^-)}_{=0}$$

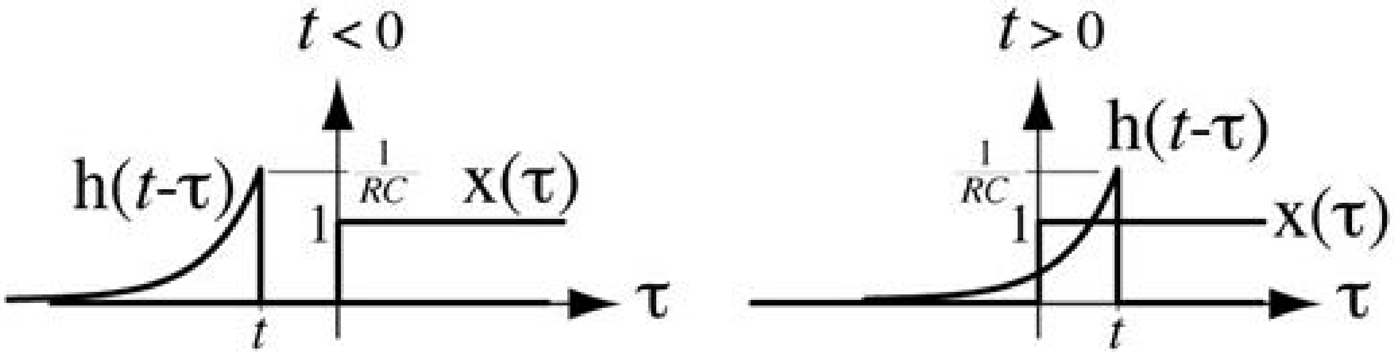
$$K+3K\left[\frac{e^{-3t}}{-3}\right]_0^{0^+}=K+3K\underbrace{\left[(-1/3)-(-1/3)\right]}_{=0}=1$$

$$K=1\Rightarrow \mathrm{h}(t)=e^{-3t}\,\mathrm{u}(t)$$

A Graphical Illustration of the Convolution Integral



$$v_{out}(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} u(\tau)\,\frac{e^{-(t-\tau)/RC}}{RC}\,u(t-\tau)\,d\tau$$



A Graphical Illustration of the Convolution Integral

$$t < 0: \quad \mathrm{v}_{out}(t) = 0$$

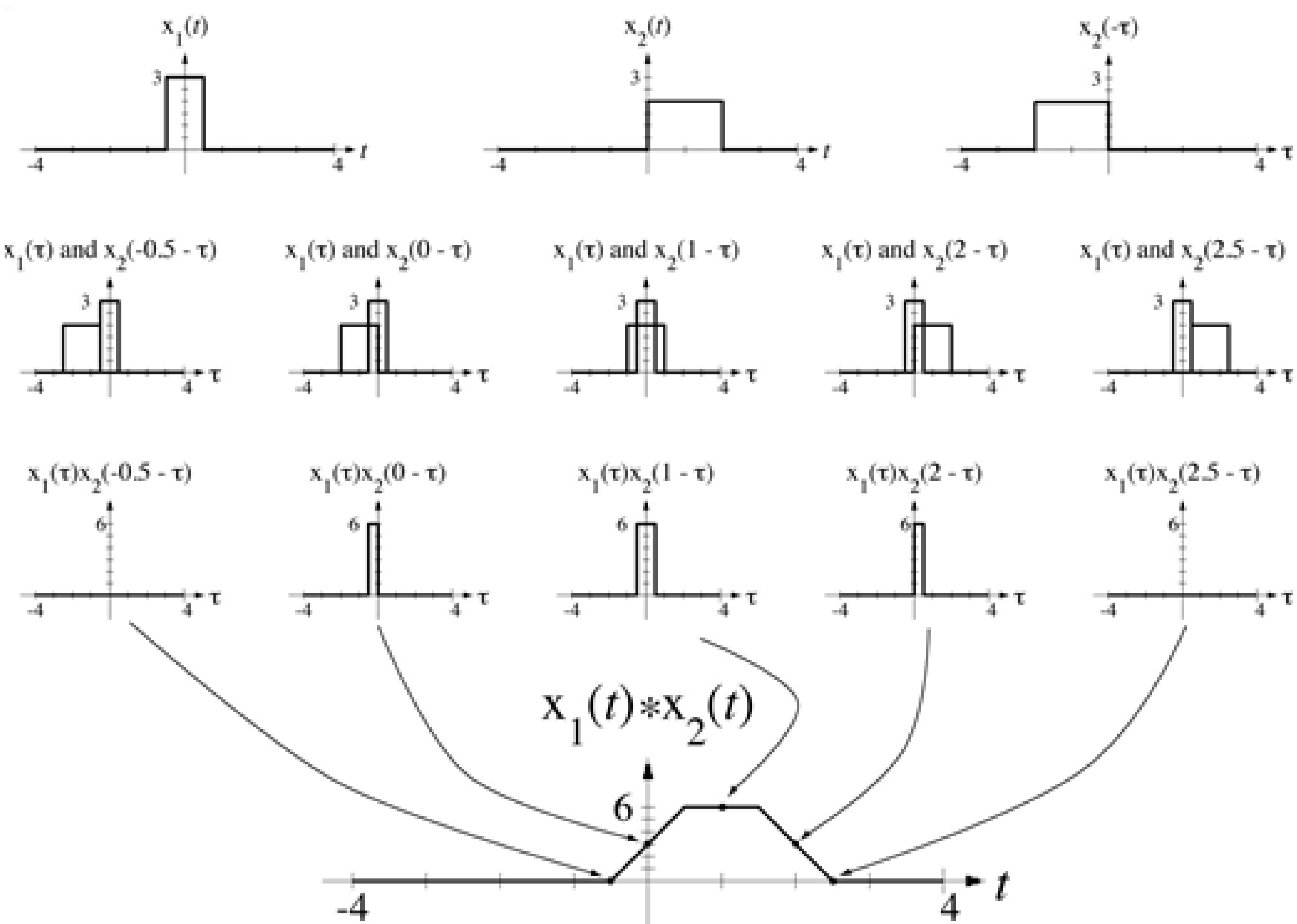
$$t > 0: \quad \mathrm{v}_{out}(t) = \int_{-\infty}^{\infty} \mathrm{u}(\tau)\frac{e^{-(t-\tau)/RC}}{RC}\mathrm{u}(t-\tau)\,d\tau$$

$$\mathrm{v}_{out}(t) = \frac{1}{RC}\int_{0}^{t} e^{-(t-\tau)/RC}\,d\tau = \frac{1}{RC}\left[\frac{e^{-(t-\tau)/RC}}{-1/RC}\right]_0^t = \left[-e^{-(t-\tau)/RC}\right]_0^t = 1 - e^{-t/RC}$$

For all time, t:

$$\mathrm{v}_{out}(t) = \left(1 - e^{-t/RC}\right)\mathrm{u}(t)$$

Convolution Example



Convolution Integral Properties

$$\mathrm{x}(t) * A\delta(t-t_0) = A\,\mathrm{x}(t-t_0)$$

$$\text{If } \mathrm{g}(t) = \mathrm{g}_0(t) * \delta(t) \text{ then } \mathrm{g}(t-t_0) = \mathrm{g}_0(t-t_0) * \delta(t) = \mathrm{g}_0(t) * \delta(t-t_0)$$

$$\text{If } \mathrm{y}(t) = \mathrm{x}(t) * \mathrm{h}(t) \text{ then } \mathrm{y}'(t) = \mathrm{x}'(t) * \mathrm{h}(t) = \mathrm{x}(t) * \mathrm{h}'(t)$$

$$\text{and } \mathrm{y}(at) = |a|\,\mathrm{x}(at) * \mathrm{h}(at)$$

Commutativity

$$\mathrm{x}(t) * \mathrm{y}(t) = \mathrm{y}(t) * \mathrm{x}(t)$$

Associativity

$$[\mathrm{x}(t) * \mathrm{y}(t)] * \mathrm{z}(t) = \mathrm{x}(t) * [\mathrm{y}(t) * \mathrm{z}(t)]$$

Distributivity

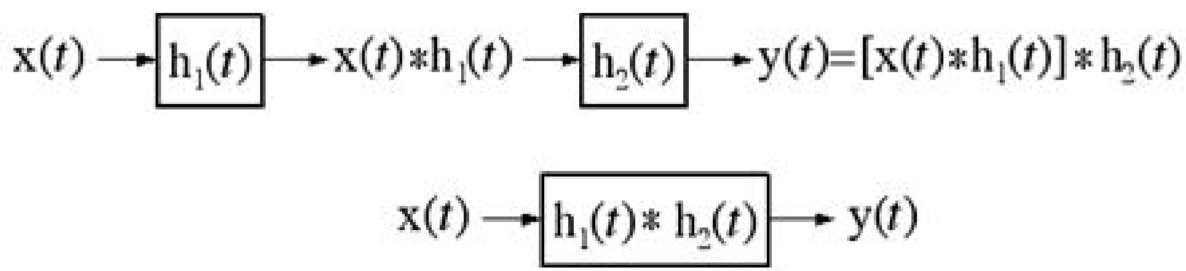
$$[\mathrm{x}(t) + \mathrm{y}(t)] * \mathrm{z}(t) = \mathrm{x}(t) * \mathrm{z}(t) + \mathrm{y}(t) * \mathrm{z}(t)$$

Cascade Connection of Systems

If the output signal from a system is the input signal to a second system the systems are said to be **cascade** connected.

It follows from the associative property of convolution that the impulse response of a cascade connection of LTI systems is the convolution of the individual impulse responses of those systems.

Cascade Connection

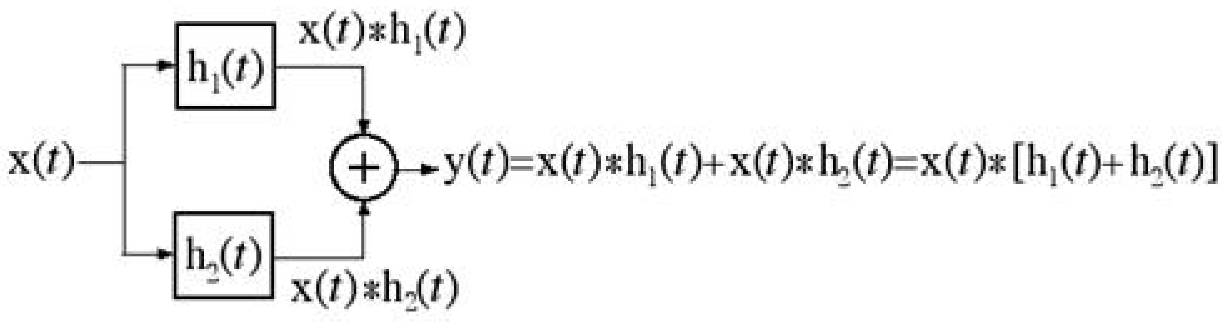


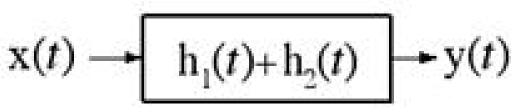
Parallel Connection Of Systems

If two systems are excited by the same signal and their responses are added they are said to be **parallel** connected.

It follows from the distributive property of convolution that the impulse response of a parallel connection of LTI systems is the sum of the individual impulse responses.

Parallel Connection





Stability and Impulse Response

A system is BIBO stable if its impulse response is **absolutely integrable**. That is if

$$\int_{-\infty}^{\infty} \left|\mathrm{h}(t)\right| dt \text{ is finite.}$$

Systems Described by Differential Equations

The most general form of a differential equation describing an LTI system is $\sum_{k=0}^{N} a_k \, \mathrm{y}^{(k)}(t) = \sum_{k=0}^{M} b_k \, \mathrm{x}^{(k)}(t)$. Let $\mathrm{x}(t) = Xe^{st}$ and let $\mathrm{y}(t) = Ye^{st}$. Then $\mathrm{x}^{(k)}(t) = s^k Xe^{st}$ and $\mathrm{y}^{(k)}(t) = s^k Ye^{st}$ and

$$\sum_{k=0}^{N} a_k s^k Ye^{st} = \sum_{k=0}^{M} b_k s^k Xe^{st}.$$

The differential equation has become an algebraic equation.

$$Ye^{st}\sum_{k=0}^{N} a_k s^k = Xe^{st}\sum_{k=0}^{M} b_k s^k \Rightarrow \frac{Y}{X} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}$$

The transfer function for systems of this type is

$$\mathrm{H}(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{b_M s^M + b_{M-1} s^{M-1} + \cdots + b_2 s^2 + b_1 s + b_0}{a_N s^N + a_{N-1} s^{N-1} + \cdots + a_2 s^2 + a_1 s + a_0}$$

This type of function is called a **rational function** because it is a ratio of polynomials in s. The transfer function encapsulates all the system characteristics and is of great importance in signal and system analysis.

Systems Described by Differential Equations

Now let $\mathrm{x}(t) = Xe^{j\omega t}$ and let $\mathrm{y}(t) = Ye^{j\omega t}$.

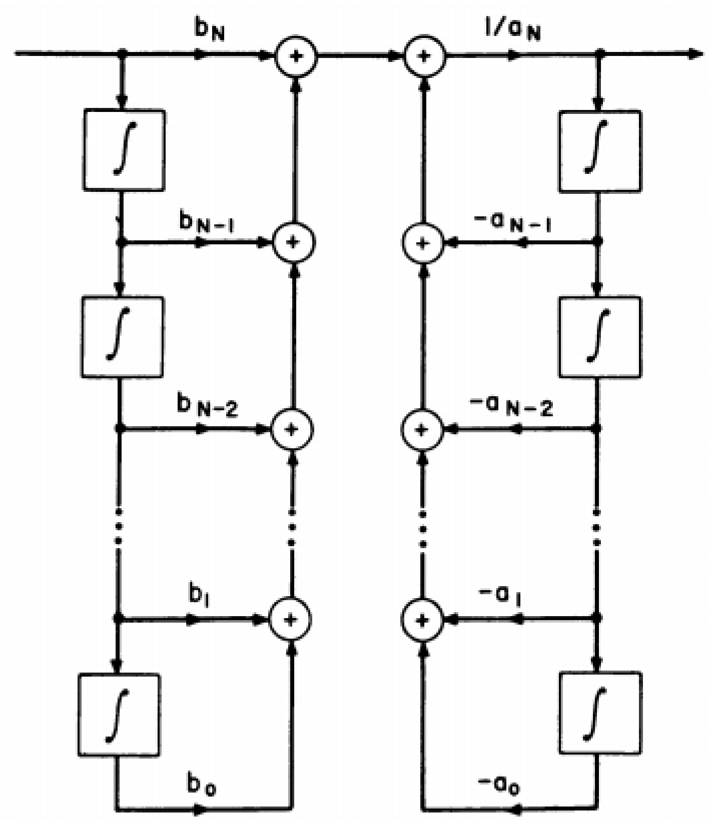
This change of variable $s \to j\omega$ changes the transfer function to the **frequency response**.

$$\mathrm{H}(j\omega) = \frac{b_M (j\omega)^M + b_{M-1}(j\omega)^{M-1} + \cdots + b_2 (j\omega)^2 + b_1 (j\omega) + b_0}{a_N (j\omega)^N + a_{N-1}(j\omega)^{N-1} + \cdots + a_2 (j\omega)^2 + a_1 (j\omega) + a_0}$$

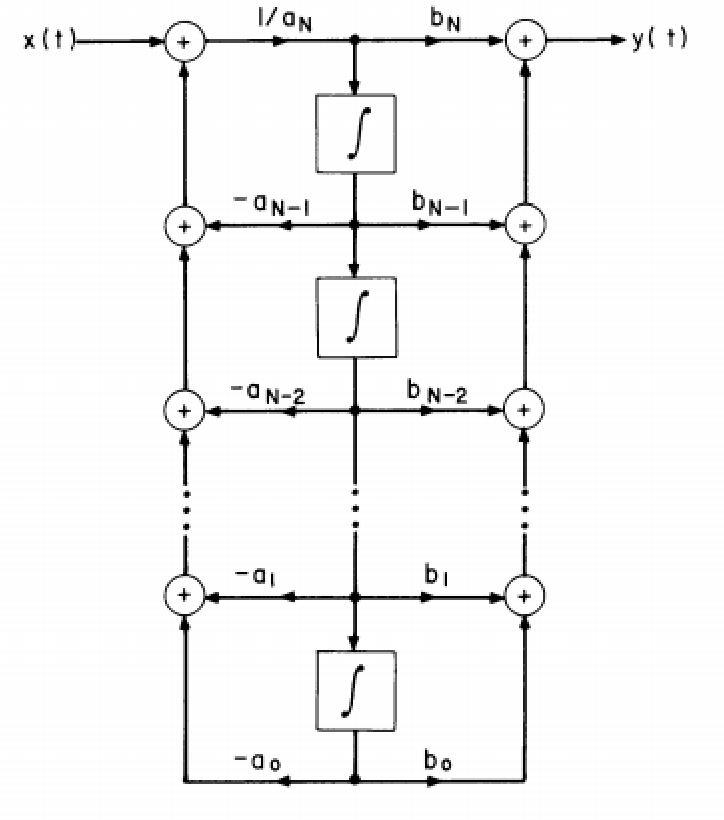
Frequency response describes how a system responds to a sinusoidal excitation, as a function of the frequency of that excitation.

It is shown in the text that if an LTI system is excited by a sinusoid $\mathrm{x}(t) = A_{\mathrm{x}} \cos(\omega_0 t + \theta_{\mathrm{x}})$ that the response is $\mathrm{y}(t) = A_{\mathrm{y}} \cos(\omega_0 t + \theta_{\mathrm{y}})$ where $A_{\mathrm{y}} = |\mathrm{H}(j\omega_0)| A_{\mathrm{x}}$ and $\theta_{\mathrm{y}} = \angle \mathrm{H}(j\omega_0) + \theta_{\mathrm{x}}$.

Block Diagram-Direct Form-I Realization



Block Diagram-Direct Form-II Realization



System Analysis using Fourier Transform

Consider the general system,

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

Our objective is to determine h(t) and H(jω). Applying CTFT on both sides:

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}.$$

Therefore, by linearity and differentiation property, we have

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega$$

The convolution property gives Y(jω) = X(jω)H(jω), so

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}.$$

we can apply the technique of partial fraction expansion to express H(jω) in a form that allows us to determine h(t).

UNIT IV
ANALYSIS OF D.T. SIGNALS

Z-Transform

The definition of the z-Transform is

$$X(z) = \sum_{k=-\infty}^{+\infty} x(k) z^{-k}.$$

where, $x(k)$ is a discrete time sequence (sampled data). When $x(k)$ is defined for $k \geq 0$, i.e. causal, one sided z-transform is given by

$$X(z) = \sum_{k=0}^{+\infty} x(k) z^{-k}.$$

The variable z is complex, so is $X(z)$.

z-Transform of simple functions

δ function

$$\delta(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{Z}[\delta(k)] = \sum_{k=0}^{+\infty} \delta(k) z^{-k} = 1$$

unit step fu

$$u(k) = \begin{cases} 1 & \text{if } k \geq 0 \\ 0 & \text{if } k < 0 \end{cases}$$

$$\mathcal{Z}[u(k)] = \sum_{k=0}^{+\infty} u(k) z^{-k} = z^0 + z^{-1} + z^{-2} + +z^{-3} \cdots$$

Inverse Z-Transform

The inversion integral is

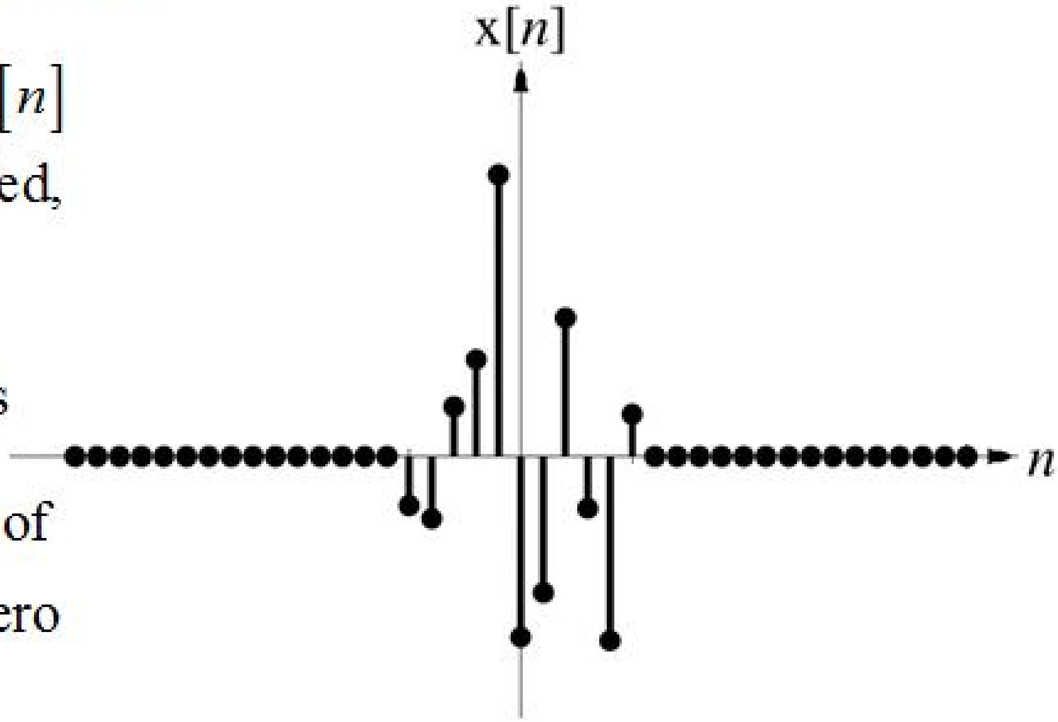
$$\mathrm{x}[n] = \frac{1}{j2\pi} \oint_C \mathrm{X}(z) z^{n-1} dz.$$

This is a contour integral in the complex plane and is beyond the scope of this course. The notation $\mathrm{x}[n] \overset{\mathcal{Z}}{\longleftrightarrow} \mathrm{X}(z)$ indicates that $\mathrm{x}[n]$ and $\mathrm{X}(z)$ form a "z-transform pair".

Existence of the z -Transform

Time Limited Signals

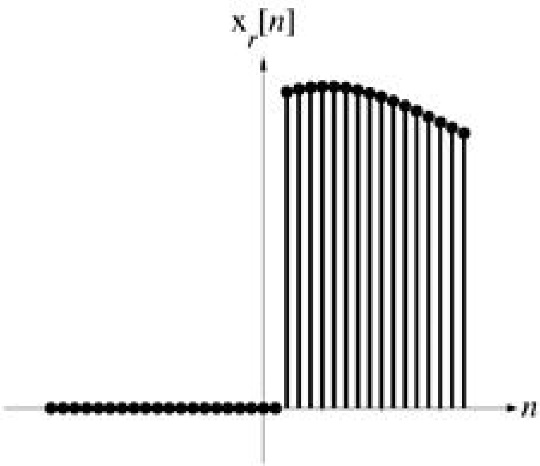
If a discrete-time signal $\mathrm{x}[n]$ is time limited and bounded, the z transformation summation $\sum_{n=-\infty}^{\infty} \mathrm{x}[n] z^{-n}$ is finite and the z transform of $\mathrm{x}[n]$ exists for any non-zero value of z.

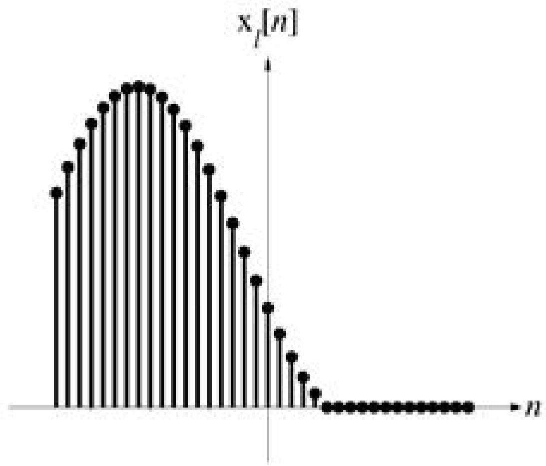


Existence of the *z* Transform

Right- and Left-Sided Signals

A right-sided signal $x_r[n]$ is one for which $x_r[n]=0$ for any $n<n_0$ and a left-sided signal $x_l[n]$ is one for which $x_l[n]=0$ for any $n>n_0$.





Region of Convergence

The Region of Convergence (ROC) of the z-transform is the set of z such that X(z) converges, i.e.,

X(z) exists if an $\sum_{n=-\infty}^{\infty} |x[n]| r^{-n} < \infty.$ nt z is inside the Region of Convergence in the z plane.

ROC is very important in analyzing the system stability and behavior

The z-transform exists for signals that do not have DTFT.

ROC: $|z| > |a| \Longleftrightarrow$ causal system

ROC: $|z| < |a| \Longleftrightarrow$ anti-causal system

ROC: $|b| < |z| < |a| \Longleftrightarrow$ two-sided system (non-causal)

ROC includes $|z|=1$ i.e. the unit circle $\Longleftrightarrow$ stable system

Properties of ROC

Property 1: The ROC is a ring or disk in the z-plane centre at origin

Property 2: DTFT of x[n] exists if and only if ROC includes the unit circle

Property 3:The ROC contains no poles.

Property 4:If x[n] is a finite impulse response (FIR), then the ROC is the entire z-plane.

Property 5: If x[n] is a right-sided sequence, then ROC extends outward from the outermost pole.

Property 6: If x[n] is a left-sided sequence, then ROC extends inward from the innermost pole.

Property 7: If X(z) is rational, i.e., X(z) = A(z) B(z) where A(z) and B(z) are polynomials, and if x[n] is right-sided, then the ROC is the region outside the outermost pole

Examples with ROC

Example 1: The Z transform of a right sided signal $x[n] = a^n u[n]$ is

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

For this summation to converge, i.e., for X(z) to exist, it is necessary to have $|az^{-1}| < 1$, i.e., the ROC is |z|>|a|.

As a special case when a=1 $x[n] = u[n]$ and we have

$$\mathcal{Z}[u[n]] = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

Examples with ROC (contd..)

Example 2: The Z-transform of a left sided signal

$x[n] = -a^n u[-n-1]$

$$
\begin{aligned}
X(z) &= -\sum_{n=-\infty}^{\infty} a^n u[-n-1] z^{-n} = -\sum_{n=-\infty}^{-1} (az^{-1})^n \\
&= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = 1 - \frac{1}{1-a^{-1}z} = \frac{z}{z-a} = \frac{1}{1-az^{-1}}
\end{aligned}
$$

For the summation above to converge, it is required $|a^{-1}z| < 1$ that , i.e., the ROC is |z|< |a| . Comparing the two examples above we see that two different signals can have identical z-transform, but with different ROCs.

Some Common z Transform Pairs

$$\delta[n] \xleftrightarrow{z} 1 \quad , \ \text{All } z$$

$$\mathrm{u}[n] \xleftrightarrow{z} \frac{z}{z-1} = \frac{1}{1-z^{-1}} \ , \ |z| > 1 \quad , \quad -\mathrm{u}[-n-1] \xleftrightarrow{z} \frac{z}{z-1} \ , \ |z| < 1$$

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{z} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \ , \ |z| > |\alpha| \quad , \quad -\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{z} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \ , \ |z| < |\alpha|$$

$$n\,\mathrm{u}[n] \xleftrightarrow{z} \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2} \ , \ |z| > 1 \quad , \quad -n\,\mathrm{u}[-n-1] \xleftrightarrow{z} \frac{z}{(z-1)^2} = \frac{z^{-1}}{(1-z^{-1})^2} \ , \ |z| < 1$$

$$n\alpha^n \mathrm{u}[n] \xleftrightarrow{z} \frac{\alpha z}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \ , \ |z| > |\alpha| \quad , \quad -n\alpha^n \mathrm{u}[-n-1] \xleftrightarrow{z} \frac{\alpha z}{(z-\alpha)^2} = \frac{\alpha z^{-1}}{(1-\alpha z^{-1})^2} \ , \ |z| < |\alpha|$$

$$\sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z\sin(\Omega_0)}{z^2-2z\cos(\Omega_0)+1} \ , \ |z| > 1 \quad , \quad -\sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{z} \frac{z\sin(\Omega_0)}{z^2-2z\cos(\Omega_0)+1} \ , \ |z| < 1$$

$$\cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z[z-\cos(\Omega_0)]}{z^2-2z\cos(\Omega_0)+1} \ , \ |z| > 1 \quad , \quad -\cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{z} = \frac{z[z-\cos(\Omega_0)]}{z^2-2z\cos(\Omega_0)+1} \ , \ |z| < 1$$

$$\alpha^n \sin(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z\alpha\sin(\Omega_0)}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \ , \ |z| > |\alpha| \ , \ -\alpha^n \sin(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{z} \frac{z\alpha\sin(\Omega_0)}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \ , \ |z| < |\alpha|$$

$$\alpha^n \cos(\Omega_0 n)\mathrm{u}[n] \xleftrightarrow{z} \frac{z[z-\alpha\cos(\Omega_0)]}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \ , \ |z| > |\alpha| \ , \ -\alpha^n \cos(\Omega_0 n)\mathrm{u}[-n-1] \xleftrightarrow{z} \frac{z[z-\alpha\cos(\Omega_0)]}{z^2-2\alpha z\cos(\Omega_0)+\alpha^2} \ , \ |z| < |\alpha|$$

$$\alpha^{|n|} \xleftrightarrow{z} \frac{z}{z-\alpha} - \frac{z}{z-\alpha^{-1}} \ , \ |\alpha| < |z| < |\alpha^{-1}|$$

$$\mathrm{u}[n-n_0] - \mathrm{u}[n-n_1] \xleftrightarrow{z} \frac{z}{z-1}\left(z^{-n_0} - z^{-n_1}\right) = \frac{z^{n_1-n_0-1} + z^{n_1-n_0-2} + \cdots + z + 1}{z^{n_1-1}} \ , \ |z| > 0$$

z-Transform Properties

Given the z-transform pairs $\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z)$ and $\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)$ with ROC's of $\mathrm{ROC_G}$ and $\mathrm{ROC_H}$ respectively the following properties apply to the z transform.

Linearity

$$\alpha\,\mathrm{g}[n] + \beta\,\mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \alpha\,\mathrm{G}(z) + \beta\,\mathrm{H}(z)$$

$$\mathrm{ROC} = \mathrm{ROC_G} \cap \mathrm{ROC_H}$$

Time Shifting

$$\mathrm{g}[n - n_0] \xleftrightarrow{\mathcal{Z}} z^{-n_0}\,\mathrm{G}(z)$$

$\mathrm{ROC} = \mathrm{ROC_G}$ except perhaps $z = 0$ or $z \to \infty$

Change of Scale in *z*

$$\alpha^n\,\mathrm{g}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{G}(z/\alpha)$$

$$\mathrm{ROC} = |\alpha|\,\mathrm{ROC_G}$$

z-Transform Properties

Time Reversal

$$g[-n] \xleftrightarrow{Z} G\left(z^{-1}\right)$$

$$\text{ROC} = 1/\text{ROC}_G$$

Time Expansion

$$\begin{cases} g[n/k] & , \ n/k \text{ and integer} \\ 0 & , \ \text{otherwise} \end{cases} \xleftrightarrow{Z} G\left(z^k\right)$$

$$\text{ROC} = \left(\text{ROC}_G\right)^{1/k}$$

Conjugation

$$g^*[n] \xleftrightarrow{Z} G^*\left(z^*\right)$$

$$\text{ROC} = \text{ROC}_G$$

z-Domain Differentiation

$$-n\,g[n] \xleftrightarrow{Z} z\frac{d}{dz}G(z)$$

$$\text{ROC} = \text{ROC}_G$$

z-Transform Properties

Convolution $\mathrm{g}[n] * \mathrm{h}[n] \xleftrightarrow{\mathcal{Z}} \mathrm{H}(z)\mathrm{G}(z)$

First Backward Difference $\mathrm{g}[n] - \mathrm{g}[n-1] \xleftrightarrow{\mathcal{Z}} \left(1 - z^{-1}\right)\mathrm{G}(z)$

$\mathrm{ROC} \supseteq \mathrm{ROC}_{\mathrm{G}} \cap |z| > 0$

Accumulation $\sum_{m=-\infty}^{n} \mathrm{g}[m] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-1}\mathrm{G}(z)$

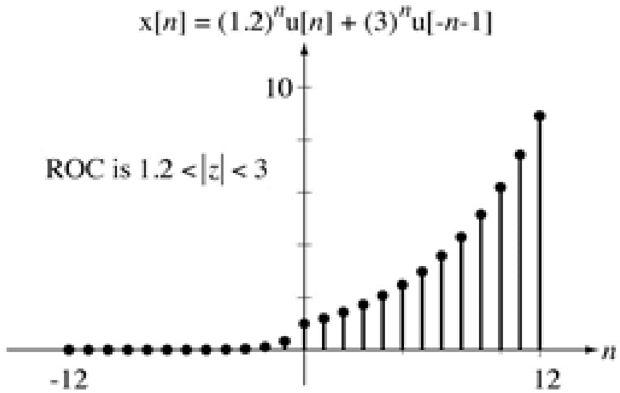
$\mathrm{ROC} \supseteq \mathrm{ROC}_{\mathrm{G}} \cap |z| > 1$

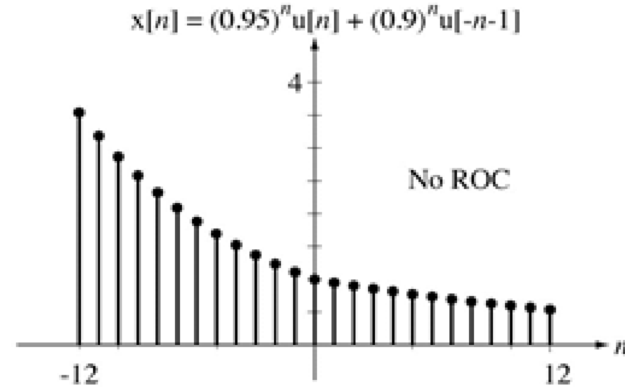
Initial Value Theorem If $\mathrm{g}[n] = 0 \ , n < 0$ then $\mathrm{g}[0] = \lim_{z \to \infty} \mathrm{G}(z)$

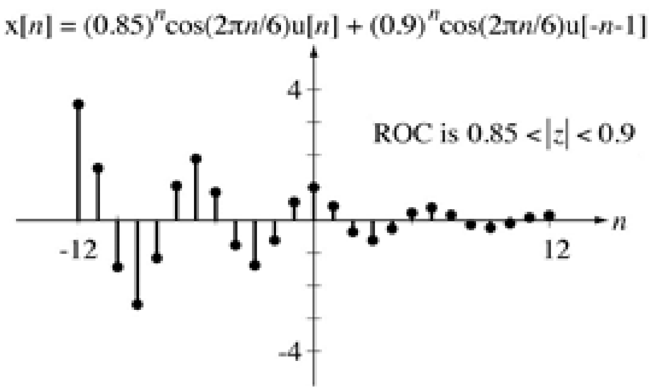
Final Value Theorem If $\mathrm{g}[n] = 0 \ , n < 0, \ \lim_{n \to \infty} \mathrm{g}[n] = \lim_{z \to 1} (z-1)\mathrm{G}(z)$

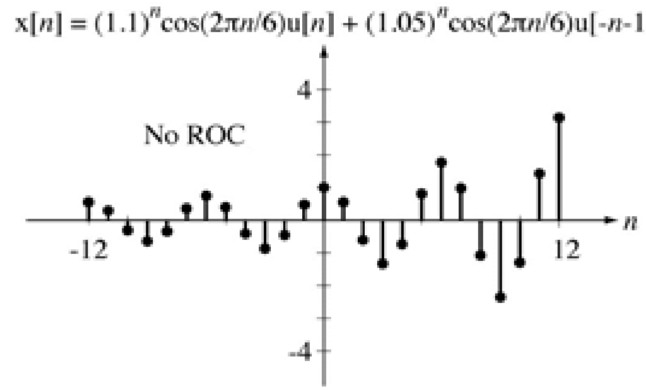
if $\lim_{n \to \infty} \mathrm{g}[n]$ exists.

z Transform -Examples









The Inverse *z* Transform

Synthetic Division

For rational *z* transforms of the form

$$\mathrm{H}(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

we can always find the inverse *z* transform by synthetic division. For example,

$$\mathrm{H}(z) = \frac{(z-1.2)(z+0.7)(z+0.4)}{(z-0.2)(z-0.8)(z+0.5)} \quad , \quad |z| > 0.8$$

$$\mathrm{H}(z) = \frac{z^3 - 0.1z^2 - 1.04z - 0.336}{z^3 - 0.5z^2 - 0.34z + 0.08} \quad , \quad |z| > 0.8$$

The Inverse z Transform

Synthetic Division

$$
\begin{array}{r}
1+0.4z^{-1}+0.5z^{-2}\cdots \\
z^3-0.5z^2-0.34z+0.08\overline{\smash{\big)}\,z^3-0.1z^2-1.04z-0.336} \\
\underline{z^3-0.5z^2-0.34z+0.08} \\
0.4z^2-\ 0.7z-\ 0.256 \\
\underline{0.4z^2\ -0.2z-\ 0.136-0.032z^{-1}} \\
0.5z-\ 0.12+0.032z^{-1} \\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

The inverse z transform is

$$\delta[n]+0.4\delta[n-1]+0.5\delta[n-2]\cdots \xleftrightarrow{\mathcal{Z}} 1+0.4z^{-1}+0.5z^{-2}\cdots$$

The Inverse *z* Transform

Synthetic Division

We could have done the synthetic division this way.

$$
\begin{array}{r}
-4.2 - 30.85z - 158.613z^2 \cdots \\
0.08 - 0.34z - 0.5z^2 + z^3 \overline{\big)\, -0.336 - 1.04z - 0.1z^2 + z^3} \\
\underline{-0.336 + 1.428z + 2.1z^2 - 4.2z^3} \\
-2.468z - 2.2z^2 + 5.2z^3 \\
\underline{-2.468z + 10.489z^2 + 15.425z^3 - 30.85z^4} \\
-12.689z^2 - 10.225z^3 + 30.85z^4 \\
\vdots \qquad \vdots \qquad \vdots
\end{array}
$$

$$-4.2\delta[n] - 30.85\delta[n+1] - 158.613\delta[n+2]\cdots \xleftrightarrow{\mathcal{Z}} -4.2 - 30.85z - 158.613z^2 \cdots$$

but with the restriction $|z| > 0.8$ this second form does not converge and is therefore not the inverse z transform.

Partial Fraction Expansion

Partial-fraction expansion works for inverse z transforms the same way it does for inverse Laplace transforms. But there is a situation that is quite common in inverse z transforms which deserves mention. It is very common to have z-domain functions in which the number of finite zeros equals the number of finite poles (making the expression improper in z) with at least one zero at $z = 0$.

$$\mathrm{H}(z) = \frac{z^{N-M}(z - z_1)(z - z_2)\cdots(z - z_M)}{(z - p_1)(z - p_2)\cdots(z - p_N)}$$

Partial Fraction Expansion

Dividing both sides by z we get

$$\frac{\mathrm{H}(z)}{z} = \frac{z^{N-M-1}(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

and the fraction on the right is now proper in z and can be expanded in partial fractions.

$$\frac{\mathrm{H}(z)}{z} = \frac{K_1}{z-p_1} + \frac{K_2}{z-p_2} + \cdots + \frac{K_N}{z-p_N}$$

Then both sides can be multiplied by z and the inverse transform can be found.

$$\mathrm{H}(z) = \frac{zK_1}{z-p_1} + \frac{zK_2}{z-p_2} + \cdots + \frac{zK_N}{z-p_N}$$

$$\mathrm{h}[n] = K_1 p_1^n \,\mathrm{u}[n] + K_2 p_2^n \,\mathrm{u}[n] + \cdots + K_N p_N^n \,\mathrm{u}[n]$$

Inverse z -Transform Example-1

Find the inverse z transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad 0.5 < |z| < 2$$

Right-sided signals have ROC's that are outside a circle and left-sided signals have ROC's that are inside a circle. Using

$$\alpha^n \, u[n] \xleftrightarrow{\;z\;} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

$$-\alpha^n \, u[-n-1] \xleftrightarrow{\;z\;} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| < |\alpha|$$

We get

$$(0.5)^n \, u[n] + (-2)^n \, u[-n-1] \xleftrightarrow{\;z\;} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad 0.5 < |z| < 2$$

Inverse z -Transform Example-2

Find the inverse z transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

In this case, both signals are right sided. Then using

$$\alpha^n \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| > |\alpha|$$

We get

$$\left[(0.5)^n - (-2)^n\right] \mathrm{u}[n] \xleftrightarrow{\mathcal{Z}} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| > 2$$

Inverse z Transform Example-3

Find the inverse z transform of

$$X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

In this case, both signals are left sided. Then using

$$-\alpha^n \, u[-n-1] \xleftrightarrow{z} \frac{z}{z-\alpha} = \frac{1}{1-\alpha z^{-1}} \quad , \quad |z| < |\alpha|$$

We get

$$-\left[(0.5)^n - (-2)^n\right] u[-n-1] \xleftrightarrow{z} X(z) = \frac{z}{z-0.5} - \frac{z}{z+2} \quad , \quad |z| < 0.5$$

The Unilateral Z -Transform

Definition:

$$X(z) = \sum_{n=0}^{\infty} x[n] z^{-n}$$

- *The unilateral z-transform ignores x[−1], x[−2], . . . and, hence, is typically only used for sequences that are zero for n < 0 (sometimes called causal sequences).*
- *If x[n] = 0 for all n < 0 then the unilateral and bilateral transforms are identical.*
- *Linear.*
- *No need to specify the ROC (extends outward from largest pole).*
- *Inverse z-transform is unique (right-sided).*
- *Can handle non-zero initial conditions.*

Discrete Time Fourier Transform

The discrete-time Fourier transform (DTFT) or the Fourier transform of a discrete–time sequence x[n] is a representation of the sequence in terms of the complex exponential sequence $e^{j\omega n}$

The DTFT sequence x[n] is given by

$$X(\omega) = \Sigma_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

•*Inverse Discrete-Time Fourier Transform*

$$x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega$$

Properties of DTFT

Periodicity :

Linearity :

Time shift :

Phase shift :

Conjugacy :

Time Reversal :

Differentiation :

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$$

$$ax_1[n] + bx_2[n] \longleftrightarrow aX_1(e^{j\omega}) + bX_2(e^{j\omega})$$

$$x[n - n_0] \longleftrightarrow e^{-j\omega n_0} X(e^{j\omega})$$

$$e^{j\omega_0 n} x[n] \longleftrightarrow X(e^{j(\omega-\omega_0)})$$

$$x^*[n] \longleftrightarrow X^*(e^{-j\omega})$$

$$x[-n] \longleftrightarrow X(e^{-j\omega})$$

$$nx[n] \longleftrightarrow j\frac{dX(e^{j\omega})}{d\omega}$$

Properties of DTFT

Parseval Equality :

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

Convolution :

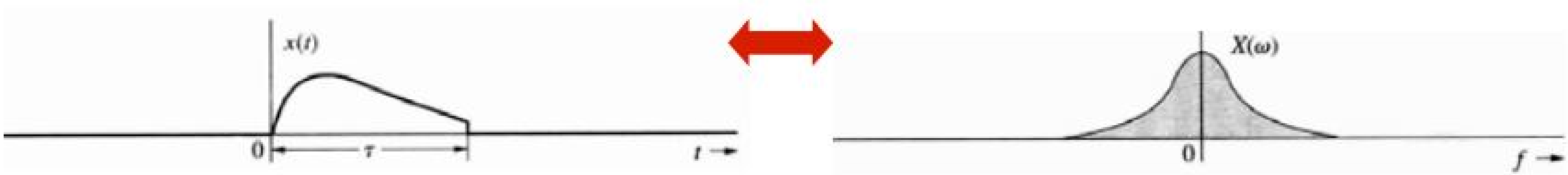
$$y[n] = x[n] * h[n] \longleftrightarrow Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega})$$

Multiplication :

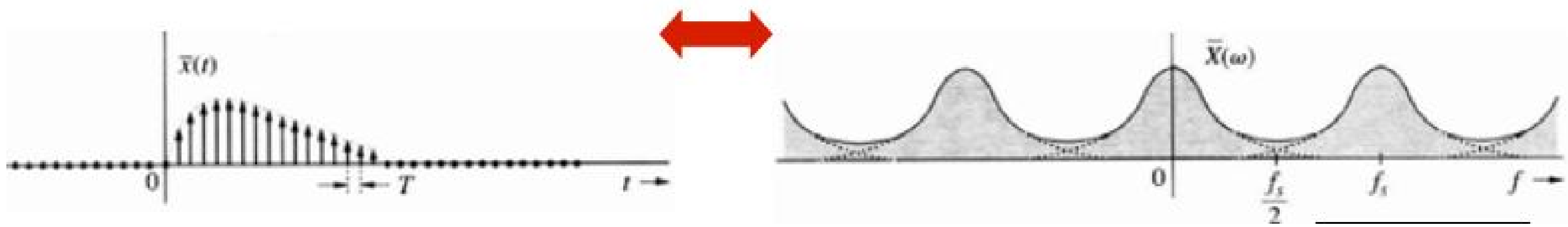
$$y[n] = x_1[n]x_2[n] \longleftrightarrow Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta$$

Discrete Fourier Transform (DFT)

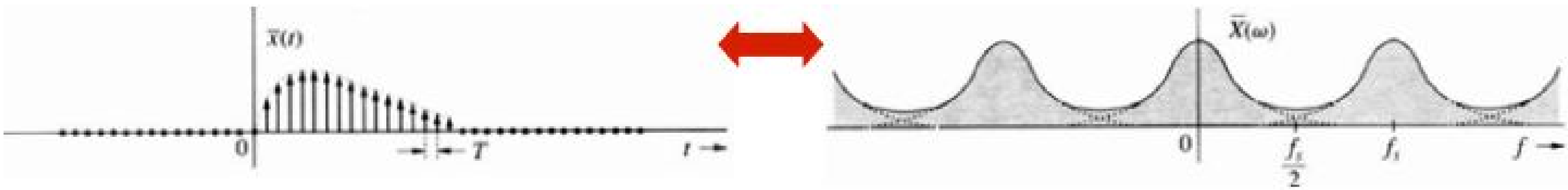
Fourier transform is computed (on computers) using discrete techniques. Such numerical computation of the Fourier transform is known as Discrete Fourier Transform (DFT). Begin with time-limited signal x(t), we want to compute its Fourier Transform X(ω).



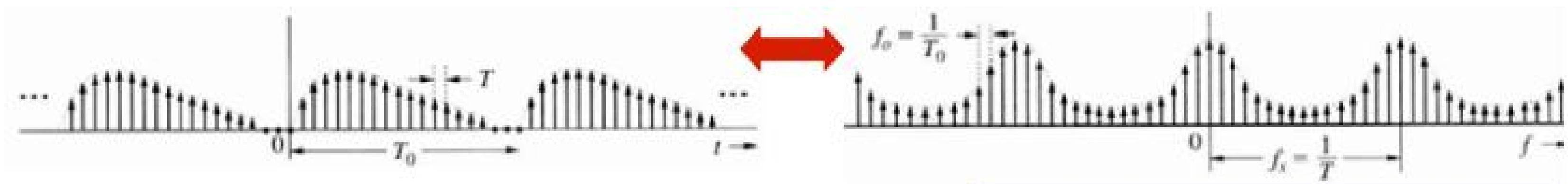
We know the effect of sampling in time domain:



Discrete Fourier Transform (DFT) (2)



•*Now construct the sampled version of x(t) as repeated copies. The effect is sampling the spectrum.*



•*Number of time samples in T0*

$$N_0 = \frac{T_0}{T}$$

Formal definition of DFT

If x(nT) and X($r\omega_0$) are the nth and rth samples of x(t) and X(ω) respectively, then we define:

and

$$x_n = T x(nT) = \frac{T_0}{N_0} x(nT)$$

$$X_r = X(r\omega_0)$$

where

$$\omega_0 = 2\pi f_0 = \frac{2\pi}{T_0}$$

Then Forv

$$X_r = \sum_{n=0}^{N_0-1} x_n e^{-jr\Omega_0 n}$$

Backward DFT :

$$x_n = \frac{1}{N_0} \sum_{r=0}^{N_0-1} X_r e^{jr\Omega_0 n}$$

Properties Of Discrete Fourier Transform

Let $\mathcal{F}[x[m]] = X(e^{j\omega})$ and $\mathcal{F}[y[m]] = Y(e^{j\omega})$ then,

Linearity :

$$\mathcal{F}[ax[m] + by[m]] = aX(e^{j\omega}) + bY(e^{j\omega})$$

Time Shifting :

$$\mathcal{F}[x[m - m_0]] = e^{-jm_0\omega}X(e^{j\omega})$$

Time Reversal :

$$\mathcal{F}[x[-m]] = X(e^{-j\omega})$$

Frequency Shifting :

$$\mathcal{F}[x[m]e^{j\omega_0 m}] = X(e^{j(\omega-\omega_0)})$$

Differencing :

$$\mathcal{F}[x[m] - x[m-1]] = (1 - e^{-j\omega})X(e^{j\omega})$$

Differentiation in frequency :

$$\mathcal{F}^{-1}[j\frac{d}{d\omega}X(e^{j\omega})] = m\ x[m]$$

Properties Of Discrete Fourier Transform

Convolution Theorems : The convolution theorem states that convolution in time domain corresponds to multiplication in frequency domain and vice versa.

------- (1)

$$\mathcal{F}[x[n] * y[n]] = X(e^{j\omega})\, Y(e^{j\omega})$$

------- (2)

$$\mathcal{F}[x[n]\; y[n]] = X(e^{j\omega}) * Y(e^{j\omega})$$

Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_0^{2\pi} |X(e^{j\omega})|^2 d\omega$$

UNIT V

LTI-DT SYSTEMS

Linear Constant Co-efficient Difference Equation

$$\sum_{k=0}^{N} a_k \, y[n-k] = \sum_{k=0}^{M} b_k \, x[n-k] \qquad (1)$$

$$\sum_{k=0}^{N} a_k \, y_h[n-k] = 0 \quad \text{(Homogeneous Equation)} \qquad (2)$$

If $y_p[n]$ satisfies (1) then so does

$y_p[n] + y_h[n]$ where $y_h[n]$ satisfies (2)

$y_p[n] \triangleq$ Particular solution

$y_h[n] \triangleq$ Homogeneous solution

Homogenous Solution

$$\sum_{k=0}^{N} a_k \, y_h[n-k] = 0$$

"guess" solution of the form

$$y_h[n] = Az^n$$

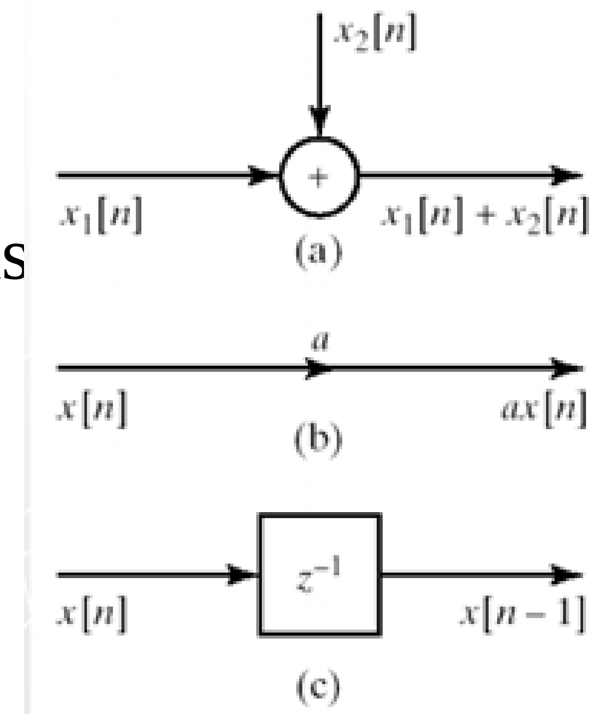
$$\sum_{k=0}^{N} a_k \, A z^n z^{-k} = 0$$

$$\sum_{k=0}^{N} a_k \, z^{-k} = 0 \quad \text{N roots } z_1, z_2, \ldots, z_N$$

$$y_h[n] = A_1 \, z_1{}^n + \ldots + A_N \, z_N{}^n$$

Block Diagram Representation

LTI systems with rational system functions can be represented as Linear constant co-efficient difference equations The implementation of difference equations requires delayed values of the sample.



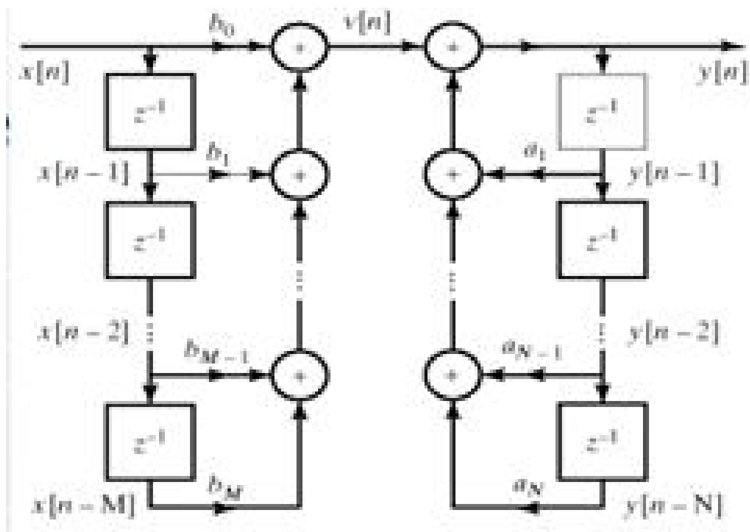
Direct Form-I Realization

General form of Difference Equation

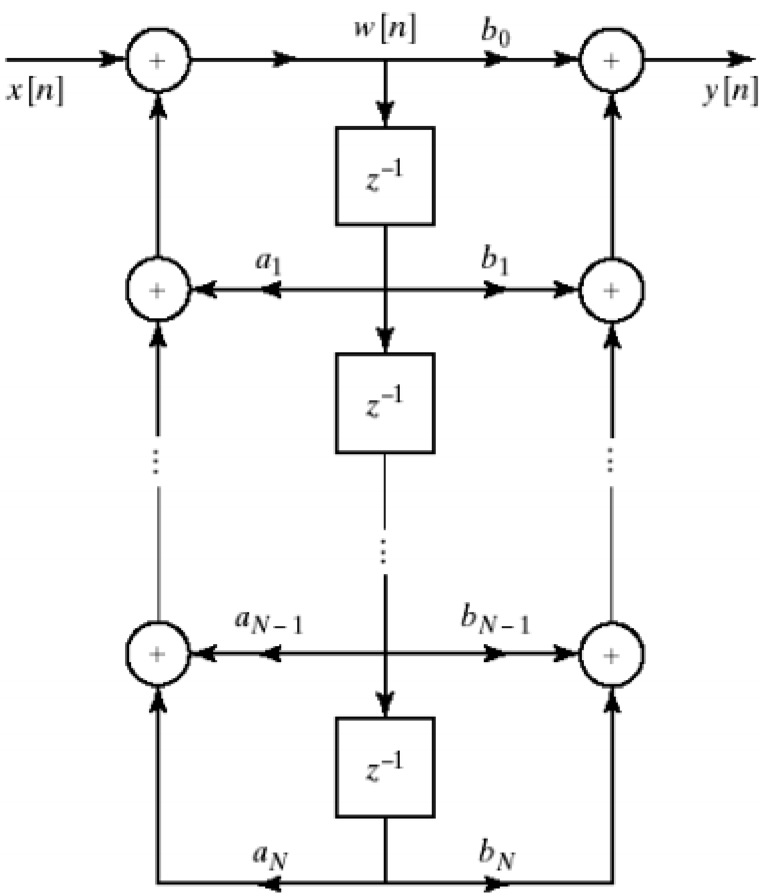
Re-arranging ,

$$\sum_{k=0}^{N} \hat{a}_k y[n-k] = \sum_{k=0}^{M} \hat{b}_k x[n-k]$$

$$y[n] - \sum_{k=1}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$



Block Diagram-Direct Form-II Realization



- No need to store the same data twice in previous system

- So we can collapse the delay elements into one chain .This is called Direct Form II or the Canonical Form

-Theoretically no difference between Direct Form I and II

-Implementation wise

i. Less memory in Direct II

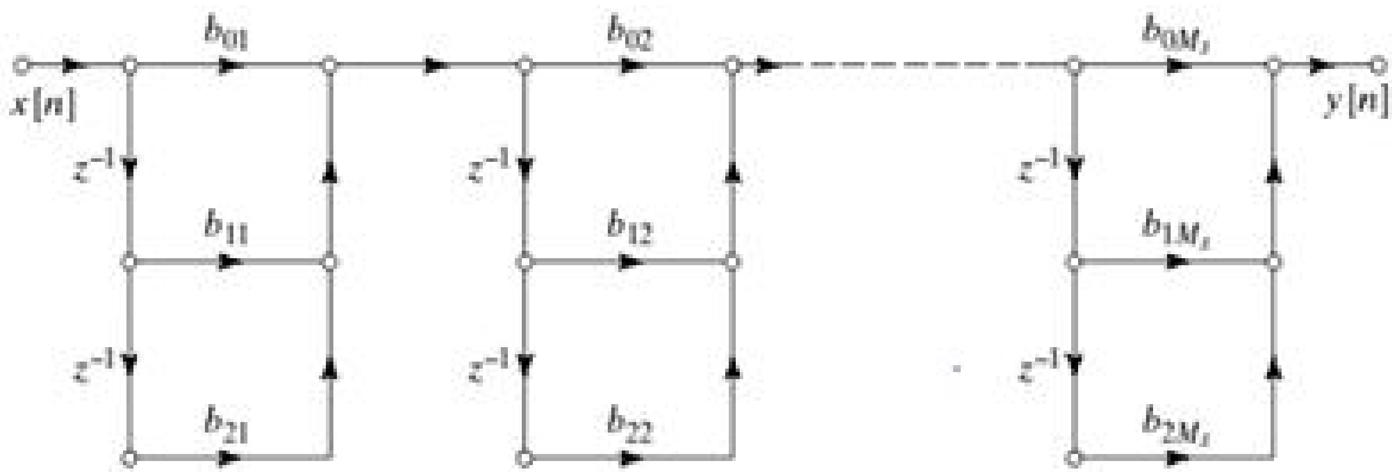
ii. Difference when using finite

\- precision arithmetic.

Cascade form of Realization

Obtained by factoring the polynomial system function.

$$H(z) = \sum_{n=0}^{M} h[n] z^{-n} = \prod_{k=1}^{M_s} \left(b_{0k} + b_{1k} z^{-1} + b_{2k} z^{-2} \right)$$

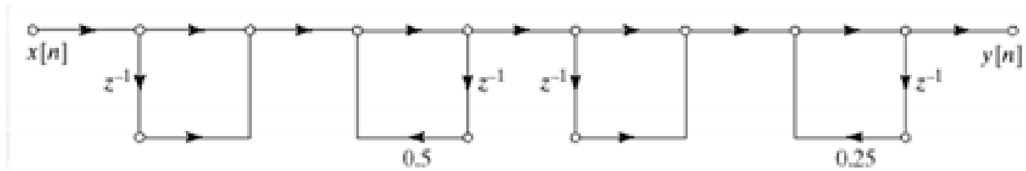


Cascade Form -Example

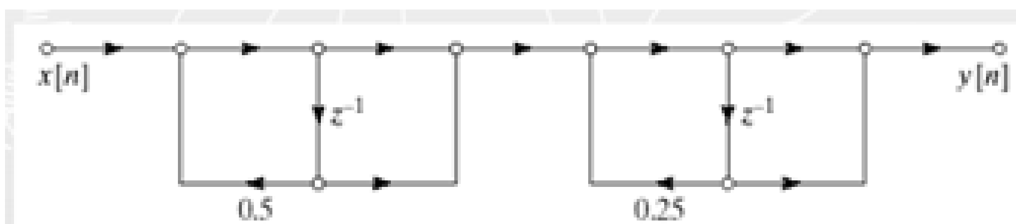
$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 - 0.25z^{-1})}$$

$$= \frac{(1 + z^{-1})}{(1 - 0.5z^{-1})} \frac{(1 + z^{-1})}{(1 - 0.25z^{-1})}$$

Cascade of Direct Form I subsections



Cascade of Direct Form II subsections



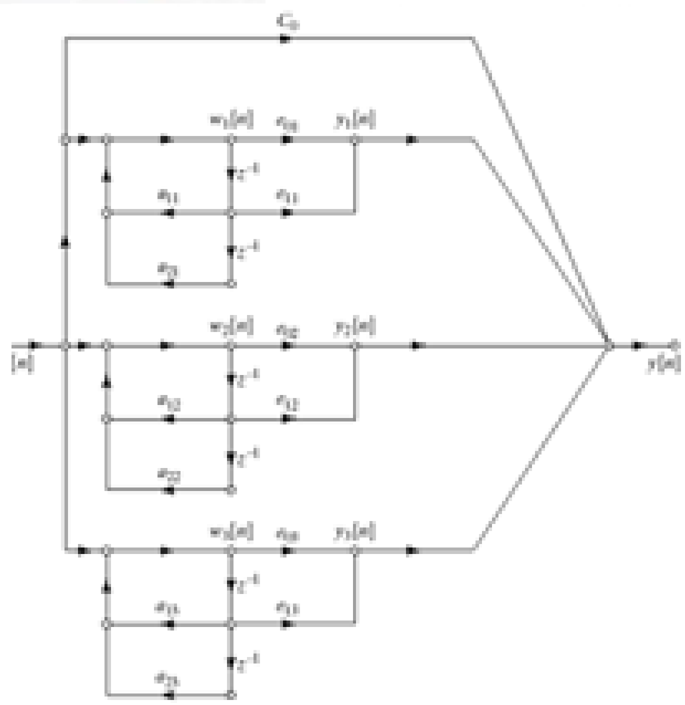
Parallel Form of Realization

Represent system function using partial fraction expansion

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_p} \frac{A_k}{1 - c_k z^{-1}} + \sum_{k=1}^{N_p} \frac{B_k\left(1 - e_k z^{-1}\right)}{\left(1 - d_k z^{-1}\right)\left(1 - d_k^{*} z^{-1}\right)}$$

$$H(z) = \sum_{k=0}^{N_p} C_k z^{-k} + \sum_{k=1}^{N_S} \frac{e_{0k} + e_{1k} z^{-1}}{1 - a_{1k} z^{-1} - a_{2k} z^{-2}}$$

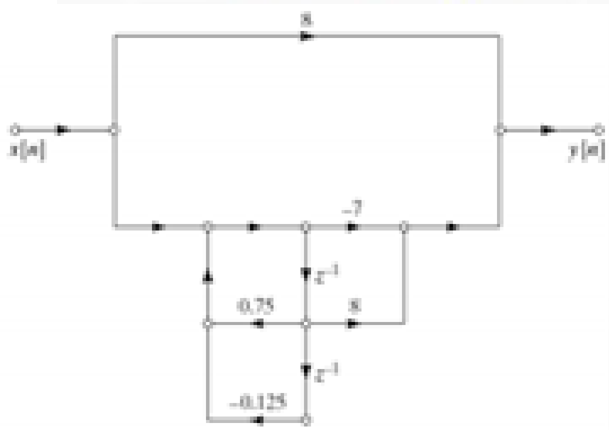
Or by pairingthe real poles



Parallel Form-Example

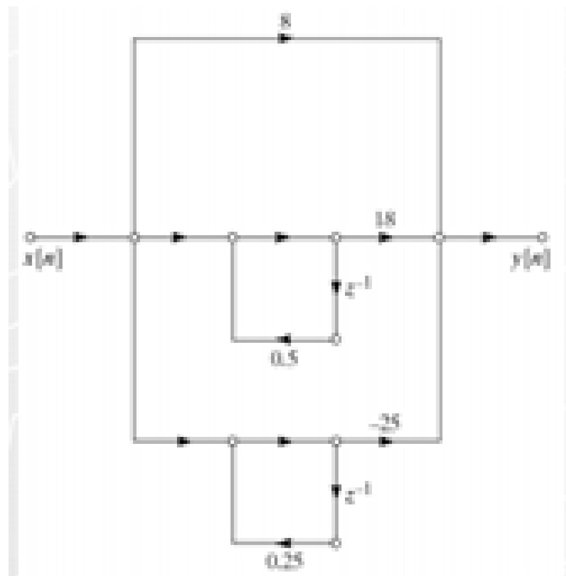
Partial Fraction Expansion

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}} = 8 + \frac{18}{(1 - 0.5z^{-1})} - \frac{25}{(1 - 0.25z^{-1})}$$



Combine poles to get

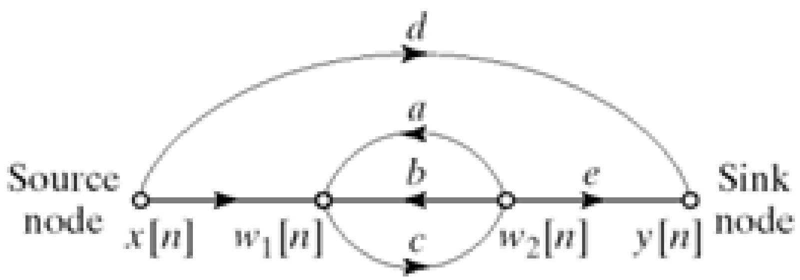
$$H(z) = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

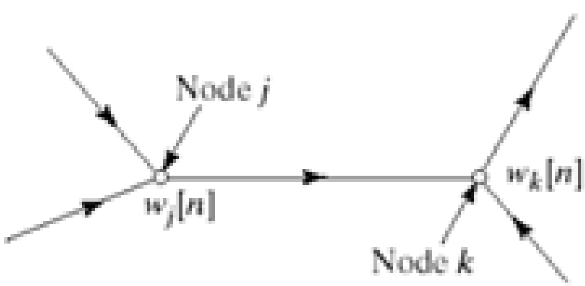


Signal Flow Graph Representation

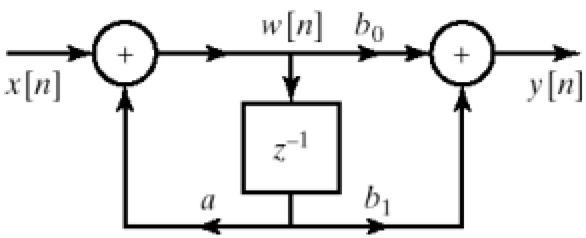
Similar to block diagram representation

A network of directed branches connected at nodes.

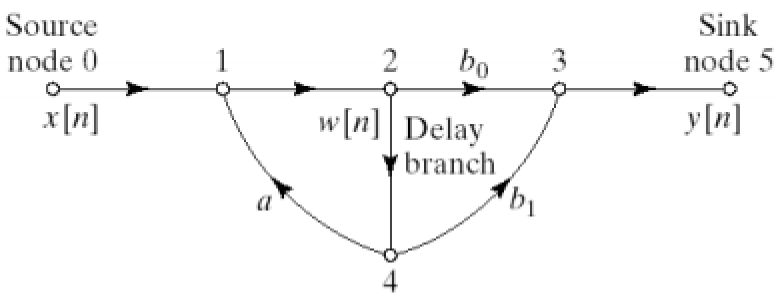


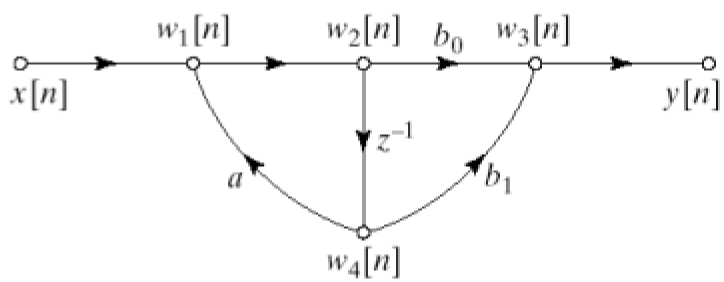


SFG-Example



(a)





$$w_1[n] = aw_4[n] + x[n]$$

$$w_2[n] = w_1[n]$$

$$w_3[n] = b_0 w_2[n] + b_1 w_4[n]$$

$$w_4[n] = w_2[n-1]$$

$$y[n] = w_3[n]$$

$$w_1[n] = aw_1[n-1] + x[n]$$

$$y[n] = b_0 w_1[n] + b_1 w_1[n-1]$$

System Properties using z-transform

CAUSALITY

Property 1. A discrete-time LTI system is causal if and only if ROC is the exterior of a circle (including ∞).

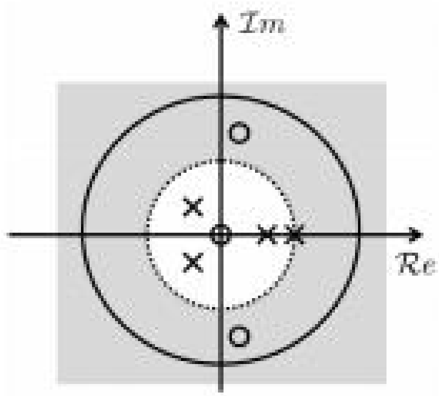
STABILITY

Property 2. A discrete-time LTI system is stable if and only if ROC of $H(z)$ includes the unit circle.

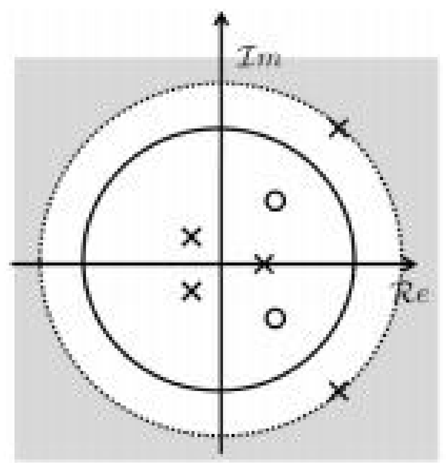
Property 3. A causal discrete-time LTI system is stable if and only if all of its poles are inside the unit circle.

System Properties using z-transform

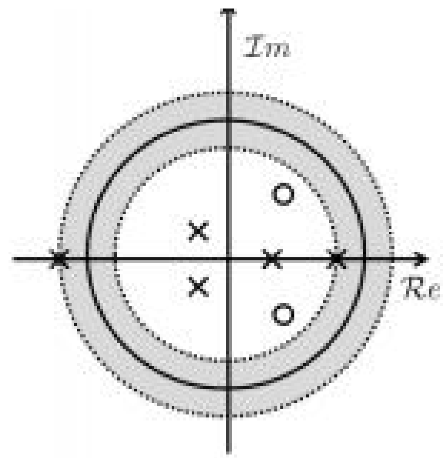
Examples



Causal, Stable



Causal, Unstable



Not causal, Stable